Chapter 4

CONTINUOUS-TIME SYSTEM ANALYSIS USING THE LAPLACE TRANSFORM

THE LAPLACE TRANSFORM

For a signal x(t), its Laplace transform X(s) is defined by:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

where $s = \sigma + j\omega$ is a complex number called the complex frequency.

Note:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t} dt$$

So, Laplace transform is similar to finding the Fourier transform of the signal $\{x(t)e^{-\sigma t}\}$. If $\sigma = 0$, Fourier transform will be the same as Laplace transform.

Example:

Find the Laplace Transform of the time function: $x(t) = Ae^{-at}u(t)$

Solution:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$X(s) = \int_{0}^{\infty} Ae^{-at}e^{-st}dt$$

$$X(s) = \int_{0}^{\infty} Ae^{-(s+a)t}dt$$

$$X(s) = A\left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}$$

$$X(s) = A\left[\frac{e^{-\infty}}{-(s+a)} - \frac{e^{-0}}{-(s+a)}\right]$$

$$\boxed{X(s) = \frac{A}{s+a}}$$

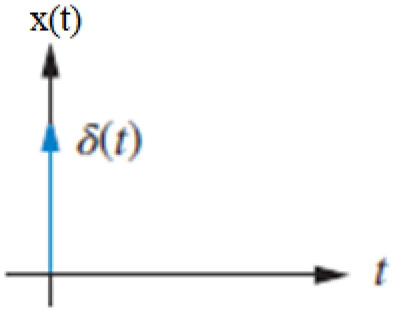
Example:

Find the Laplace Transform of the impulse function: $x(t) = \delta(t)$

Solution:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$X(s) = \int_{-\infty}^{\infty} \delta(t)e^{-st}dt$$



$$X(s) = e^{-s\times 0}\int_{-\infty}^{\infty} \delta(t)dt$$

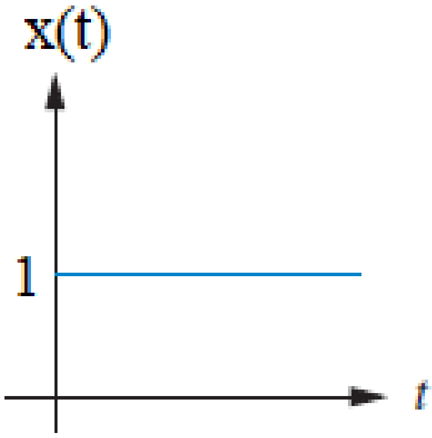
$$\boxed{X(s) = 1}$$

Example:

Find the Laplace Transform of the unit step function: $x(t) = u(t)$

Solution:

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$



$$X(s) = \int_{0}^{\infty} e^{-st} dt$$

$$X(s) = \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$X(s) = \frac{e^{-\infty}}{-s} - \frac{e^{0}}{-s}$$

$$\boxed{X(s) = \frac{1}{s}}$$

Example:

Find the Laplace Transform of the function: $x(t) = cos(\omega_0 t)u(t)$

Solution:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$X(s) = \int_{0}^{\infty} cos(\omega_0 t)e^{-st}dt$$

$$cos(\omega_0 t) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)$$

$$X(s) = \frac{1}{2}\int_{0}^{\infty} \left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)e^{-st}dt$$

Solution:

$$X(s) = \frac{1}{2}\int_0^{\infty}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)e^{-st}dt$$

$$X(s) = \frac{1}{2}\int_0^{\infty}\left(e^{-(s-j\omega_0)t} + e^{-(s+j\omega_0)t}\right)dt$$

$$X(s) = \frac{1}{2}\left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} + \frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)}\right]_0^{\infty}$$

$$X(s) = \frac{1}{2}\left[\frac{e^{-(s-j\omega_0)\infty}}{-(s-j\omega_0)} + \frac{e^{-(s+j\omega_0)\infty}}{-(s+j\omega_0)} - \left(\frac{e^{0}}{-(s-j\omega_0)} + \frac{e^{0}}{-(s+j\omega_0)}\right)\right]$$

$$X(s) = \frac{1}{2}\left[\frac{1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right]$$

$$X(s) = \frac{1}{2}\left[\frac{s-j\omega_0+s+j\omega_0}{(s-j\omega_0)(s+j\omega_0)}\right]$$

Solution:

$$X(s) = \frac{1}{2}\left[\frac{s - j\omega_0 + s + j\omega_0}{(s - j\omega_0)(s + j\omega_0)}\right]$$

$$\boxed{X(s) = \frac{s}{(s^2 + \omega_0^2)}}$$

Example:

Find the Laplace Transform of the function: $x(t) = Ate^{-at}u(t)$

Hint: $\int_0^{\infty} te^{-at}dt = \frac{1}{a^2}$

Solution:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

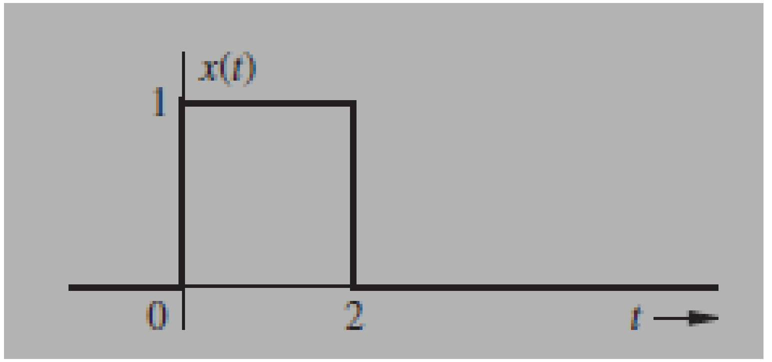
$$X(s) = \int_{0}^{\infty} Ate^{-at}e^{-st}dt$$

$$X(s) = A\int_{0}^{\infty} te^{-(s+a)t}dt$$

$$X(s) = \frac{A}{(s+a)^2}$$

Example:

Find the Laplace Transform of the function $x(t) = rect\left(\frac{t-1}{2}\right)$ shown in figure.



Solution:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$X(s) = \int_{0}^{2} 1e^{-st}dt$$

$$X(s) = \left[\frac{e^{-st}}{-s}\right]_0^2$$

$$X(s) = \frac{e^{-2s}}{-s} - \frac{e^{0}}{-s}$$

$$X(s) = \frac{e^{-2s}}{-s} + \frac{1}{s}$$

$$X(s) = \frac{1}{s}(1 - e^{-2s})$$

TABLE 4.1 Laplace Transform Pairs

No.	$x(t)$	$X(s)$
1	$\delta(t)$	1
2	$u(t)$	$\frac{1}{s}$
3	$tu(t)$	$\frac{1}{s^2}$
4	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5	$e^{\lambda t}u(t)$	$\frac{1}{s-\lambda}$
6	$te^{\lambda t}u(t)$	$\frac{1}{(s-\lambda)^2}$
7	$t^n e^{\lambda t}u(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$
8a	$\cos bt\, u(t)$	$\frac{s}{s^2+b^2}$
8b	$\sin bt\, u(t)$	$\frac{b}{s^2+b^2}$
9a	$e^{-at}\cos bt\, u(t)$	$\frac{s+a}{(s+a)^2+b^2}$
9b	$e^{-at}\sin bt\, u(t)$	$\frac{b}{(s+a)^2+b^2}$

Inverse Laplace Transform

- we can find the inverse transforms using the table shown in slide 11. All we need is to express X(s) as a sum of simpler functions of the forms listed in the table.
- Most of the transforms X(s) of practical interest are rational functions, that is, ratios of polynomials in s. Such functions can be expressed as a sum of simpler functions by using partial fraction expansion.
- Values of s at which X(s) = 0 are called the zeros of X(s); the values of s at which X(s)$\rightarrow \infty$ are called the poles of X(s).

$$X(s) = \frac{N(s)}{D(s)}$$

- The roots of N(s) are the zeros, and the roots of D(s) are the poles of X(s).

Partial-Fraction Expansion:

$$X(s) = \frac{N(s)}{D(s)}$$

- If we convert X(s) to a sum of a simpler functions using partial fraction expansion, then:

$$X(s) = X_1(s) + X_2(s) + X_3(s) + \cdots$$

- Then using inverse Laplace transform:

$$x(t) = x_1(t) + x_2(t) + x_3(t) + \cdots$$

Case 1: Roots of the denominator of X(s) are Real and Distinct

Example:

Find the inverse Laplace Transform of the function

$$X(s) = \frac{7s - 6}{(s+2)(s-3)}$$

Solution:

The roots of D(s) are s = – 2, and s = + 3

The highest power of N(s) is less than that of D(s), we can write the partial-fraction expansion as:

$$X(s) = \frac{k_1}{s+2} + \frac{k_2}{s-3}$$

Where k_1, and k_2 are constants to be determined:

To get k_1 and k_2, we use coverup Method.

$$X(s) = \frac{7s - 6}{(s + 2)(s - 3)} \qquad X(s) = \frac{k_1}{s + 2} + \frac{k_2}{s - 3}$$

To get k_1: Coverup (s+2), then substitute s = – 2

$$k_1 = \left. \frac{7s - 6}{\boxed{}(s - 3)} \right|_{s=-2} = \frac{-20}{-5} = 4$$

To get k_2: Coverup (s-3), then substitute s = 3

$$k_2 = \left. \frac{7s - 6}{(s + 2)\boxed{}} \right|_{s=3} = \frac{15}{5} = 3$$

So the function X(s) is:

$$X(s) = \frac{4}{s + 2} + \frac{3}{s - 3}$$

$$X(s) = \frac{4}{s+2} + \frac{3}{s-3}$$

- The inverse Laplace transform is:

Using the table in slide 11: $$e^{\lambda t}u(t) \rightarrow \frac{1}{s-\lambda}$$

$$x(t) = 4e^{-2t}u(t) + 3e^{3t}u(t)$$

$$x(t) = (4e^{-2t} + 3e^{3t})u(t)$$

Example:

Find the inverse Laplace Transform of the function

$$X(s) = \frac{2s^2 + 5}{s^2 + 3s + 2}$$

Solution:

$$X(s) = \frac{2s^2 + 5}{(s+2)(s+1)}$$

- **The roots of D(s) are s = – 2, and s = – 1**

We can write the partial-fraction expansion as:

$$X(s) = 2 + \frac{k_1}{s+2} + \frac{k_2}{s+1}$$

Observe that X(s) is an improper function with the power of N(s) and D(s) is the same. The constant 2 in the above equation is (Coefficient of s^2 in N(s)/Coefficient of s^2 in D(s)).

To get k_1 and k_2, we use coverup Method.

$$X(s) = \frac{2s^2+5}{(s+2)(s+1)} \qquad X(s) = 2 + \frac{k_1}{s+2} + \frac{k_2}{s+1}$$

To get k_1: Coverup (s+2), then substitute s = – 2

$$k_1 = \left.\frac{2s^2+5}{\boxed{}(s+1)}\right|_{s=-2} = \frac{13}{-1} = -13$$

To get k_2: Coverup (s+1), then substitute s = – 1

$$k_2 = \left.\frac{2s^2+5}{(s+2)\boxed{}}\right|_{s=-1} = \frac{7}{1} = 7$$

So the function X(s) is:

$$X(s) = 2 + \frac{-13}{s+2} + \frac{7}{s+1}$$

$$X(s) = 2 + \frac{-13}{s+2} + \frac{7}{s+1}$$

- The inverse Laplace transform is:

Using the table in slide 11:

$$\delta(t) \rightarrow 1$$

$$e^{\lambda t}u(t) \rightarrow \frac{1}{s-\lambda}$$

$$x(t) = 2\delta(t) - 13e^{-2t}u(t) + 7e^{-t}u(t)$$

$$x(t) = 2\delta(t) + (7e^{-t} - 13e^{-2t})u(t)$$

Case 2. Roots of the denominator of X(s) are real and repeated

Example:

Find the inverse Laplace Transform of the function

$$X(s) = \frac{8s + 10}{(s+1)(s+2)^2}$$

The roots of D(s) are s = – 1, s = – 2, and s = – 2

In this case, we can write the partial-fraction expansion as:

$$X(s) = \frac{k_1}{s+1} + \frac{k_2}{(s+2)^2} + \frac{k_3}{s+2}$$

Where k_1, k_2 , and k_3 are constants to be determines:

To get k_1 and k_2, we use coverup Method.

$$X(s) = \frac{8s + 10}{(s+1)(s+2)^2} \qquad X(s) = \frac{k_1}{s+1} + \frac{k_2}{(s+2)^2} + \frac{k_3}{s+2}$$

To get k_1: Coverup (s+1), then substitute s = – 1

$$k_1 = \left.\frac{8s + 10}{(s+2)^2}\right|_{s=-1} = \frac{2}{1} = 2$$

To get k_2: Coverup $(s+2)^2$, then substitute s = – 2

$$k_2 = \left.\frac{8s + 10}{(s+1)}\right|_{s=-2} = \frac{-6}{-1} = 6$$

So the function X(s) is:

$$X(s) = \frac{2}{s+1} + \frac{6}{(s+2)^2} + \frac{k_3}{s+2}$$

To get k_3, we have two methods:

$$X(s) = \frac{8s + 10}{(s+1)(s+2)^2} \qquad X(s) = \frac{2}{s+1} + \frac{6}{(s+2)^2} + \frac{k_3}{s+2}$$

Method #1: Multiply X(s) by $(s+2)^2$, differentiate with respect to s. then substitute s = – 2

$$k_3 = \frac{d}{ds}\left(\frac{8s+10}{s+1}\right)\bigg|_{s=-2}$$

$$k_3 = \frac{(s+1)\times 8 - (8s+10)\times 1}{(s+1)^2}\bigg|_{s=-2}$$

$$k_3 = \frac{-2}{(s+1)^2}\bigg|_{s=-2} = \frac{-2}{(-2+1)^2} = -2$$

Another Method to get k_3,

$$X(s) = \frac{8s+10}{(s+1)(s+2)^2} \qquad X(s) = \frac{2}{s+1} + \frac{6}{(s+2)^2} + \frac{k_3}{s+2}$$

Method #2: Multiply X(s) by *s*, then substitute s $\rightarrow \infty$

$$sX(s) = \frac{s(8s+10)}{(s+1)(s+2)^2} \qquad sX(s) = \frac{2s}{s+1} + \frac{6s}{(s+2)^2} + \frac{k_3 s}{s+2}$$

$$\left.\frac{s(8s+10)}{(s+1)(s+2)^2}\right|_{s\to\infty} = 0 \qquad \left.\frac{2s}{s+1} + \frac{6s}{(s+2)^2} + \frac{k_3 s}{s+2}\right|_{s\to\infty} = 2 + k_3$$

$$2 + k_3 = 0 \qquad k_3 = -2$$

$$X(s) = \frac{2}{s+1} + \frac{6}{(s+2)^2} - \frac{2}{s+2}$$

So the function X(s) is: $$X(s) = \frac{2}{s+1} + \frac{6}{(s+2)^2} - \frac{2}{s+2}$$

- The inverse Laplace transform is:

Using the table in slide 11:

$$e^{\lambda t}u(t) \rightarrow \frac{1}{s-\lambda}$$

$$te^{\lambda t}u(t) \rightarrow \frac{1}{(s-\lambda)^2}$$

$$x(t) = 2e^{-t}u(t) + 6te^{-2t}u(t) - 2e^{-2t}u(t)$$

$$x(t) = (2e^{-t} + 6te^{-2t} - 2e^{-2t})u(t)$$

Example:

Find the inverse Laplace transform of: $X(s) = \frac{10}{s(s+2)(s+3)^2}$

Solution:

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+2} + \frac{k_3}{(s+3)^2} + \frac{k_4}{s+3}$$

$$k_1 = \left.\frac{10}{(s+2)(s+3)^2}\right|_{s=0} = \frac{10}{2(3)^2} = \frac{5}{9}$$

$$k_2 = \left.\frac{10}{s(s+3)^2}\right|_{s=-2} = \frac{10}{-2(1)^2} = -5$$

$$k_3 = \left.\frac{10}{s(s+2)}\right|_{s=-3} = \frac{10}{-3(-1)} = \frac{10}{3}$$

Solution:

$$X(s) = \frac{10}{s(s+2)(s+3)^2}$$

$$X(s) = \frac{k_1}{s} + \frac{k_2}{s+2} + \frac{k_3}{(s+3)^2} + \frac{k_4}{s+3}$$

To get k_4,

Method #1:

$$k_4 = \frac{d}{ds}\left(\frac{10}{s(s+2)}\right)\Bigg|_{s=-3}$$

$$k_4 = \left(\frac{-10(s+s+2)}{s^2(s+2)^2}\right)\Bigg|_{s=-3} = \frac{-10(-6+2)}{9(-1)^2} = \frac{40}{9}$$

Another Method to get k_4,

Method #2: Multiply X(s) by *s*, then substitute s $\rightarrow$ ∞

$$X(s) = \frac{10}{s(s+2)(s+3)^2} \qquad X(s) = \frac{5}{9s} - \frac{5}{s+2} + \frac{10}{3(s+3)^2} + \frac{k_4}{s+3}$$

$$sX(s) = \frac{10}{(s+2)(s+3)^2} \qquad sX(s) = \frac{5}{9} - \frac{5s}{s+2} + \frac{10s}{3(s+3)^2} + \frac{k_4 s}{s+3}$$

$$\left.\frac{10}{(s+2)(s+3)^2}\right|_{s\to\infty} = 0 \quad \left.\frac{5}{9} - \frac{5s}{s+2} + \frac{10s}{3(s+3)^2} + \frac{k_4 s}{s+3}\right|_{s\to\infty} = \frac{5}{9} - 5 + k_4$$

$$\frac{5}{9} - 5 + k_4 = 0$$

$$k_4 = 5 - \frac{5}{9} = \frac{40}{9}$$

Solution:

$$X(s) = \frac{5/9}{s} - \frac{5}{s+2} + \frac{10/3}{(s+3)^2} + \frac{40/9}{s+3}$$

The inverse Laplace transform is: Using the table in slide 11:

$$u(t) \rightarrow \frac{1}{s} \qquad e^{\lambda t}u(t) \rightarrow \frac{1}{s-\lambda}$$

$$te^{\lambda t}u(t) \rightarrow \frac{1}{(s-\lambda)^2}$$

So the function x(t) is:

$$x(t) = \frac{5}{9}u(t) - 5e^{-2t}u(t) + \frac{10}{3}te^{-3t}u(t) + \frac{40}{9}e^{-3t}u(t)$$

Case 3. Roots of the denominator of X(s) are complex or imaginary

Example:

Find the inverse Laplace transform of: $X(s) = \dfrac{1}{s(s^2 + 2s + 5)}$

Solution:

The roots of D(s) are: s = 0, s = – 1-j2, and s = – 1+j2

We can write the partial-fraction expansion as:

$$X(s) = \frac{k_1}{s} + \frac{k_2 s + k_3}{(s^2 + 2s + 5)}$$

Where k_1, k_2 , and k_3 are constants to be determined:

Solution:

$$X(s) = \frac{1}{s(s^2+2s+5)} \qquad X(s) = \frac{k_1}{s} + \frac{k_2 s + k_3}{(s^2+2s+5)}$$

$$k_1 = \left.\frac{1}{(s^2+2s+5)}\right|_{s=0} = \frac{1}{5}$$

To get k_2 and k_3: Multiply X(s) by *s*, then substitute s $\rightarrow \infty$

$$sX(s) = \frac{1}{(s^2+2s+5)} \qquad sX(s) = \frac{1}{5} + \frac{s(k_2 s + k_3)}{(s^2+2s+5)}$$

$$\left.\frac{1}{(s^2+2s+5)}\right|_{s\to\infty} = 0 \qquad \left.\frac{1}{5} + \frac{s(k_2 s + k_3)}{(s^2+2s+5)}\right|_{s\to\infty} = \frac{1}{5} + k_2$$

$$\frac{1}{5} + k_2 = 0 \qquad k_2 = -\frac{1}{5}$$

Solution:

$$X(s) = \frac{1}{s(s^2 + 2s + 5)}$$

$$X(s) = \frac{1}{5s} + \frac{\frac{-1}{5}s + k_3}{(s^2 + 2s + 5)}$$

To get k_3: substitute s = 1

$$X(s) = \frac{1}{1(1 + 2 + 5)} = \frac{1}{8}$$

$$X(s) = \frac{1}{5} + \frac{\frac{-1}{5} + k_3}{(1 + 2 + 5)}$$

$$\frac{1}{8} = \frac{1}{5} + \frac{\frac{-1}{5} + k_3}{8}$$

$$\frac{1}{8} - \frac{1}{5} = \frac{\frac{-1}{5} + k_3}{8}$$

$$-\frac{3}{5} = \frac{-1}{5} + k_3$$

$$\frac{1}{5} - \frac{3}{5} = k_3$$

$$k_3 = -\frac{2}{5}$$

Solution:

So the function X(s) is:

$$X(s) = \frac{1}{5s} + \frac{\frac{-1}{5}s - \frac{2}{5}}{(s^2 + 2s + 5)}$$

$$X(s) = \frac{1}{5s} - \frac{1}{5}\frac{s+2}{(s^2 + 2s + 5)}$$

$$X(s) = \frac{1}{5s} - \frac{1}{5} \times \frac{s+2}{(s+1)^2 + 4}$$

$$X(s) = \frac{1}{5s} - \frac{1}{5} \times \frac{s+1+1}{(s+1)^2 + 4}$$

$$X(s) = \frac{1}{5s} - \frac{1}{5} \times \frac{s+1}{(s+1)^2 + 4} - \frac{1}{5} \times \frac{1}{(s+1)^2 + 4}$$

Solution:

The inverse Laplace transform is: Using the table in slide 11:

$$u(t) \to \frac{1}{s}$$

$$e^{-at}sin(bt)u(t) \to \frac{b}{(s+a)^2+b^2} \qquad e^{-at}cos(bt)u(t) \to \frac{s+a}{(s+a)^2+b^2}$$

$$X(s) = \frac{1}{5s} - \frac{1}{5}\frac{s+1}{(s+1)^2+4} - \frac{1}{5}\frac{1}{(s+1)^2+4}$$

$$X(s) = \frac{1}{5s} - \frac{1}{5}\frac{s+1}{(s+1)^2+4} - \frac{1}{10}\frac{2}{(s+1)^2+4}$$

$$x(t) = \frac{1}{5}u(t) - \frac{1}{5}e^{-t}cos(2t)\,u(t) - \frac{1}{10}e^{-t}\sin(2t)u(t)$$

$$x(t) = \left(\frac{1}{5} - \frac{1}{5}e^{-t}cos(2t) - \frac{1}{10}e^{-t}\sin(2t)\right)u(t)$$

Example:

Find the inverse Laplace transform of: $X(s) = \dfrac{6(s+34)}{s(s^2+10s+34)}$

Solution:

The roots of D(s) are s = 0, s = –5+j3, and s = – 5 – j3

We can write the partial-fraction expansion as:

$$X(s) = \frac{k_1}{s} + \frac{k_2 s + k_3}{(s^2+10s+34)}$$

Where k_1, k_2 , and k_3 are constants to be determined:

Solution:

$$X(s) = \frac{6(s+34)}{s(s^2+10s+34)} \qquad X(s) = \frac{k_1}{s} + \frac{k_2 s + k_3}{(s^2+10s+34)}$$

$$k_1 = \left.\frac{6(s+34)}{(s^2+10s+34)}\right|_{s=0} = 6$$

To get k_2 and k_3: Multiply X(s) by *s*, then substitute s $\rightarrow$ ∞

$$sX(s) = \frac{6(s+34)}{(s^2+10s+34)} \qquad sX(s) = 6 + \frac{s(k_2 s + k_3)}{(s^2+10s+34)}$$

$$\left.\frac{6(s+34)}{(s^2+10s+34)}\right|_{s\to\infty} = 0 \qquad \left. 6 + \frac{s(k_2 s + k_3)}{(s^2+10s+34)}\right|_{s\to\infty} = 6 + k_2$$

$$6 + k_2 = 0 \qquad k_2 = -6$$

Solution:

$$X(s) = \frac{6(s+34)}{s(s^2+10s+34)}$$

$$X(s) = \frac{6}{s} + \frac{-6s+k_3}{(s^2+10s+34)}$$

To get k_3: substitute s = 1

$$X(s) = \frac{6 \times 35}{(1+10+34)} = \frac{14}{3}$$

$$X(s) = 6 + \frac{-6+k_3}{(1+10+34)}$$

$$\frac{14}{3} = 6 + \frac{-6+k_3}{45}$$

$$-\frac{4}{3} = \frac{-6+k_3}{45}$$

$$-60 = -6 + k_3$$

$$-60 + 6 = k_3$$

$$k_3 = -54$$

Solution:

So the function X(s) is:

$$X(s) = \frac{6}{s} + \frac{-6s - 54}{(s^2 + 10s + 34)}$$

$$X(s) = \frac{6}{s} - 6\frac{s + 9}{(s^2 + 10s + 34)}$$

$$X(s) = \frac{6}{s} - 6\frac{s + 9}{(s + 5)^2 + 9}$$

$$X(s) = \frac{6}{s} - 6\frac{s + 5 + 4}{(s + 5)^2 + 9}$$

$$X(s) = \frac{6}{s} - 6\frac{s + 5}{(s + 5)^2 + 9} - 6\frac{4}{(s + 5)^2 + 9}$$

Solution:

The inverse Laplace transform is: Using the table in slide 11:

$$u(t) \rightarrow \frac{1}{s}$$

$$e^{-at}sin(bt)u(t) \rightarrow \frac{b}{(s+a)^2+b^2} \qquad e^{-at}cos(bt)u(t) \rightarrow \frac{s+a}{(s+a)^2+b^2}$$

$$X(s) = \frac{6}{s} - 6\frac{s+5}{(s+5)^2+9} - 6\frac{4}{(s+5)^2+9}$$

$$X(s) = \frac{6}{s} - 6\frac{s+5}{(s+5)^2+9} - 6 \times \frac{4}{3}\frac{3}{(s+5)^2+9}$$

$$x(t) = 6u(t) - 6e^{-5t}cos(3t)\,u(t) - 8e^{-5t}\sin(3t)u(t)$$

$$x(t) = \left(6 - 6e^{-5t}cos(3t) - 8e^{-5t}\sin(3t)\right)u(t)$$

SOME PROPERTIES OF THE LAPLACE TRANSFORM

1. LINEARITY:

The Laplace transform is linear that is if:

$$x_1(t) \rightleftharpoons X_1(s) \quad \text{and} \quad x_2(t) \rightleftharpoons X_2(s)$$

Then, for all constants a_1 and a_2, we have:

$$a_1 x_1(t) + a_2 x_2(t) \rightleftharpoons a_1 X_1(s) + a_2 X_2(s)$$

SOME PROPERTIES OF THE LAPLACE TRANSFORM

2. Time Shifting

The time shift property states that if

$$x(t)u(t) \rightleftharpoons X(s)$$

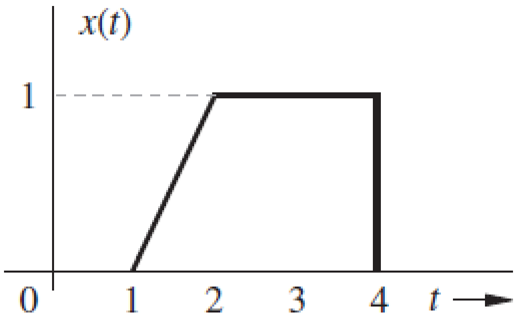
Then,

$$x(t-t_0)u(t-t_0) \rightleftharpoons X(s)e^{-st_0} \qquad t_0 \geq 0$$

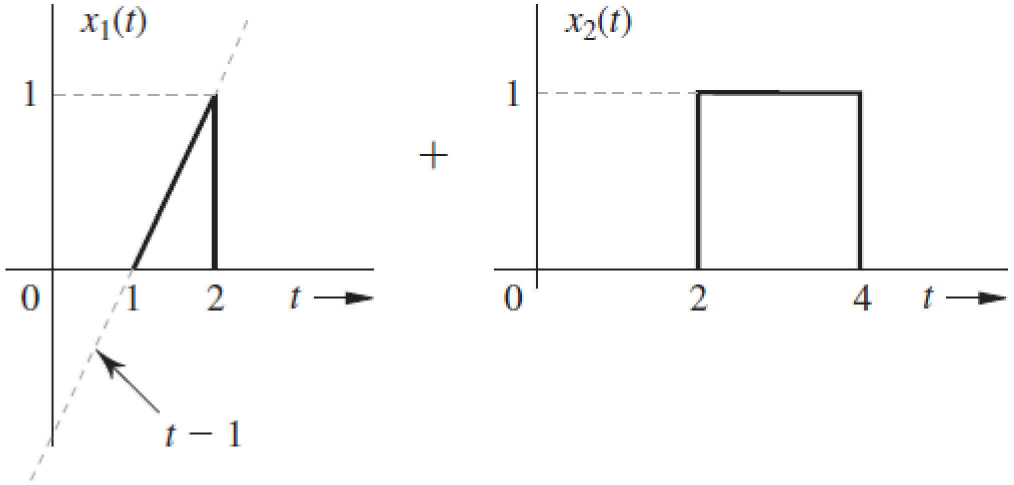
The time-shifting property proves very convenient in finding the Laplace transform of functions with different descriptions over different intervals, as the following example demonstrates.

Example:

Find the Laplace transform of x(t) depicted in figure.



Solution:



$$x(t) = x_1(t) + x_2(t)$$

$$x(t) = (t-1)[u(t-1) - u(t-2)] + [u(t-2) - u(t-4)]$$

$$x(t) = (t-1)[u(t-1) - u(t-2)] + [u(t-2) - u(t-4)]$$

$$x(t) = (t-1)u(t-1) - (t-1)u(t-2) + u(t-2) - u(t-4)$$

$$x(t) = (t-1)u(t-1) - [(t-1) - 1]u(t-2) - u(t-4)$$

$$x(t) = (t-1)u(t-1) - (t-2)u(t-2) - u(t-4)$$

From the table in slide 11:

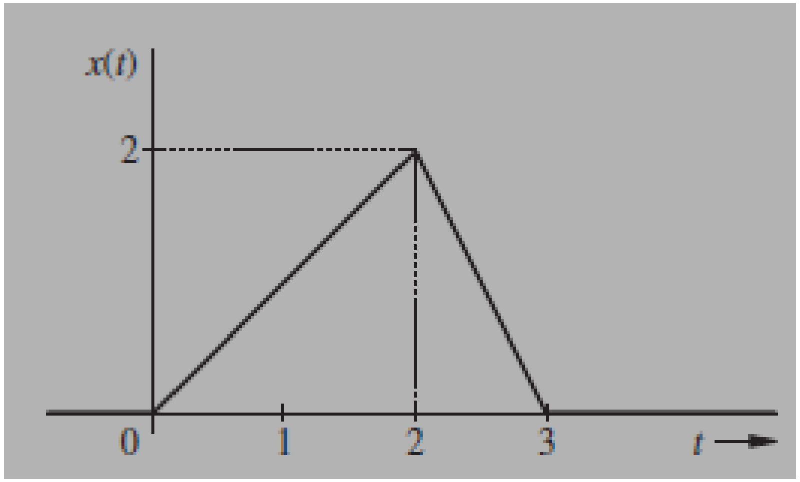
$$u(t) \to \frac{1}{s} \qquad u(t-t_0) \to \frac{1}{s}e^{-st_0}$$

$$tu(t) \to \frac{1}{s^2} \qquad (t-t_0)u(t-t_0) \to \frac{1}{s^2}e^{-st_0}$$

$$X(s) = \frac{1}{s^2}e^{-s} - \frac{1}{s^2}e^{-2s} - \frac{1}{s}e^{-4s}$$

Example:

Find the Laplace transform of x(t) depicted in figure.



Solution:

$$x(t) = x_1(t) + x_2(t)$$

$$x(t) = t[u(t) - u(t-2)] + 2(3-t)[u(t-2) - u(t-3)]$$

$$x(t) = tu(t) - tu(t-2) + 2(3-t)u(t-2) - 2(3-t)u(t-3)$$

$$x(t) = tu(t) + [2(3-t) - t]u(t-2) - 2(3-t)u(t-3)$$

$$x(t) = tu(t) + [6-3t]u(t-2) - 2(3-t)u(t-3)$$

Solution:

$$x(t) = tu(t) + [6 - 3t]u(t-2) - 2(3-t)u(t-3)$$

$$x(t) = tu(t) - 3(t-2)u(t-2) + 2(t-3)u(t-3)$$

From the table in slide 11:

$$tu(t) \rightarrow \frac{1}{s^2} \qquad (t-t_0)u(t-t_0) \rightarrow \frac{1}{s^2}e^{-st_0}$$

$$X(s) = \frac{1}{s^2} - \frac{3}{s^2}e^{-2s} + \frac{2}{s^2}e^{-3s}$$

$$X(s) = \frac{1}{s^2}(1 - 3e^{-2s} + 2e^{-3s})$$

Example:

Find the inverse Laplace Transform of the function

$$X(s) = \frac{5e^{-2s}}{(s+1)(s+2)}$$

Solution:

Let

$$X_1(s) = \frac{5}{(s+1)(s+2)}$$

- **The roots of D(s) are s = – 1, and s = – 2**

We can write the partial-fraction expansion as:

$$X_1(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

Solution:

To get k_1 and k_2, we use coverup Method.

$$X_1(s) = \frac{5}{(s+1)(s+2)} \qquad X_1(s) = \frac{k_1}{s+1} + \frac{k_2}{s+2}$$

To get k_1: Coverup (s+1), then substitute s = – 1

$$k_1 = \left.\frac{5}{(s+2)}\right|_{s=-1} = \frac{5}{1} = 5$$

To get k_2: Coverup (s+2), then substitute s = – 2

$$k_2 = \left.\frac{5}{(s+1)}\right|_{s=-2} = \frac{5}{-1} = -5$$

So the function X_1(s) is:

$$X_1(s) = \frac{5}{s+1} - \frac{5}{s+2}$$

Solution:

$$X_1(s) = \frac{5}{(s+1)(s+2)}$$

$$X(s) = \frac{5e^{-2s}}{(s+1)(s+2)}$$

$$X_1(s) = \frac{5}{s+1} - \frac{5}{s+2}$$

$$X_1(s) = \frac{5}{s+1}e^{-2s} - \frac{5}{s+2}e^{-2s}$$

The inverse Laplace transform is: Using the table in slide 11:

$$e^{\lambda t}u(t) \rightarrow \frac{1}{s-\lambda}$$

$$e^{\lambda(t-2)}u(t-2) \rightarrow \frac{1}{s-\lambda}e^{-2s}$$

$$x(t) = 5e^{-(t-2)}u(t-2) - 5e^{-2(t-2)}u(t-2)$$

$$x(t) = 5\left[e^{-(t-2)} - e^{-2(t-2)}\right]u(t-2)$$

SOME PROPERTIES OF THE LAPLACE TRANSFORM

3. Frequency Shifting

The frequency shift property states that if

$$x(t) \rightleftharpoons X(s)$$

Then,

$$x(t)e^{s_0 t} \rightleftharpoons X(s - s_0)$$

SOME PROPERTIES OF THE LAPLACE TRANSFORM

4. Time differentiation

The time-differentiation property states that if

$$x(t) \rightleftharpoons X(s)$$

Then,

$$\frac{dx(t)}{dt} \rightleftharpoons sX(s) - x(0^-)$$

$$\frac{d^2x(t)}{dt^2} \rightleftharpoons s^2X(s) - sx(0^-) - \dot{x}(0^-)$$

Generally,

$$\frac{d^nx(t)}{dt^n} \rightleftharpoons s^nX(s) - s^{n-1}x(0^-) - s^{n-2}\dot{x}(0^-) - \cdots - x^{(n-1)}(0^-)$$

SOME PROPERTIES OF THE LAPLACE TRANSFORM

5. Frequency differentiation

The frequency-differentiation property states that if

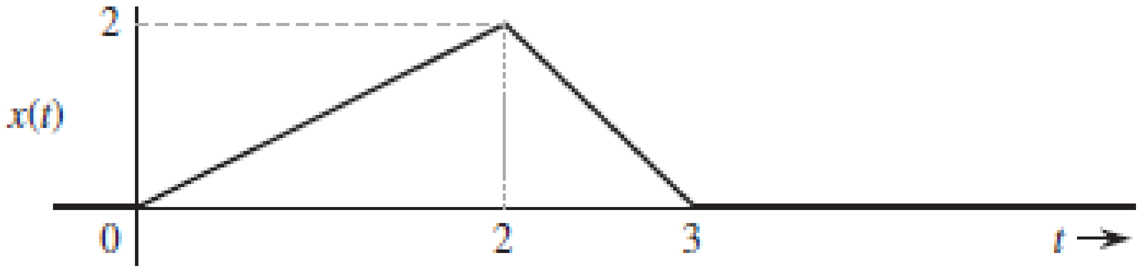
$$x(t) \rightleftharpoons X(s)$$

Then,

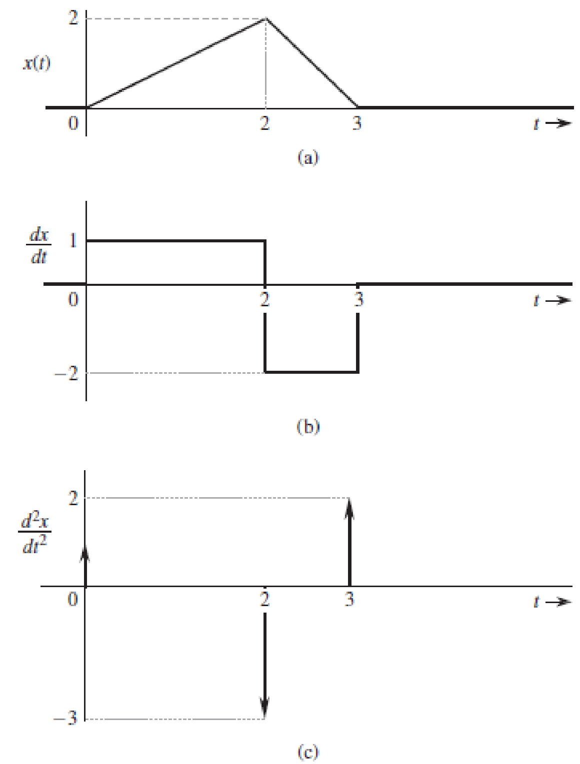
$$tx(t) \rightleftharpoons -\frac{dX(s)}{ds}$$

Example:

Find the Laplace transform of the signal x(t) using the time differentiation property and the table in slide 11.



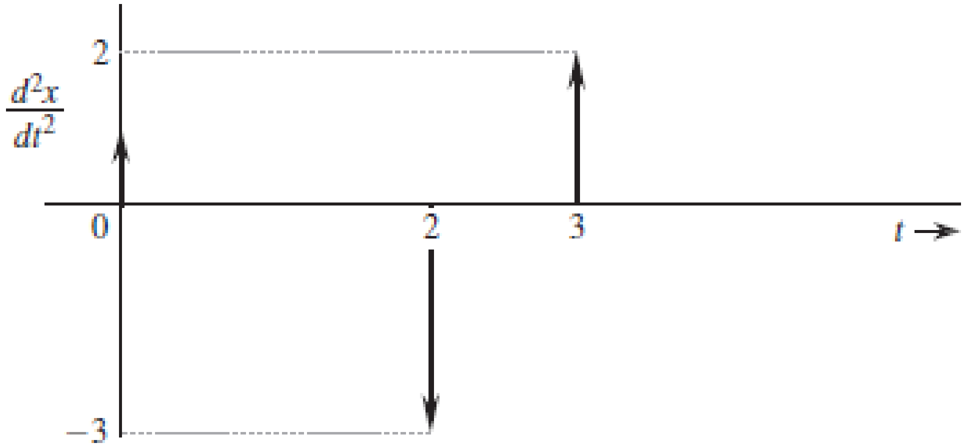
Solution:



$$\frac{d^2x(t)}{dt^2} \rightleftharpoons s^2X(s) - sx(0^-) - \dot{x}(0^-)$$

$$\frac{d^2x(t)}{dt^2} \rightleftharpoons s^2X(s)$$

Solution:



$$\frac{d^2x(t)}{dt^2} = \delta(t) - 3\delta(t-2) + 2\delta(t-3)$$

$$\frac{d^2x(t)}{dt^2} \rightleftharpoons \quad 1 - 3e^{-2s} + 2e^{-3s}$$

$$s^2X(s) = 1 - 3e^{-2s} + 2e^{-3s}$$

$$X(s) = \frac{1}{s^2}(1 - 3e^{-2s} + 2e^{-3s})$$

SOME PROPERTIES OF THE LAPLACE TRANSFORM

6. Time integration

The time integration property states that if:

$$x(t) \rightleftharpoons X(s)$$

Then,

$$\int_{0^-}^{t} x(\tau)d\tau \rightleftharpoons \frac{X(s)}{s}$$

SOME PROPERTIES OF THE LAPLACE TRANSFORM

7. Time scaling

The time scaling property states that if:

$$x(t) \rightleftharpoons X(s)$$

Then,

$$x(at) \rightleftharpoons \frac{1}{a} X\left(\frac{s}{a}\right)$$

$a > 0$

SOME PROPERTIES OF THE LAPLACE TRANSFORM

8. Time Convolution

The time convolution property states that if:

$$x_1(t) \rightleftharpoons X_1(s) \quad \text{and} \quad x_2(t) \rightleftharpoons X_2(s)$$

Then,

$$x_1(t) * x_2(t) \rightleftharpoons X_1(s)X_2(s)$$

TABLE 4.2 Unilateral Laplace Transform Properties

Operation	$x(t)$	$X(s)$
Addition	$x_1(t)+x_2(t)$	$X_1(s)+X_2(s)$
Scalar multiplication	$kx(t)$	$kX(s)$
Time differentiation	$\dfrac{dx(t)}{dt}$	$sX(s)-x(0^-)$
	$\dfrac{d^2x(t)}{dt^2}$	$s^2X(s)-sx(0^-)-\dot{x}(0^-)$
	$\dfrac{d^3x(t)}{dt^3}$	$s^3X(s)-s^2x(0^-)-s\dot{x}(0^-)-\ddot{x}(0^-)$
	$\dfrac{d^nx(t)}{dt^n}$	$s^nX(s)-\sum_{k=1}^{n} s^{n-k}x^{(k-1)}(0^-)$
Time integration	$\int_{0^-}^{t} x(\tau)\,d\tau$	$\dfrac{1}{s}X(s)$
	$\int_{-\infty}^{t} x(\tau)\,d\tau$	$\dfrac{1}{s}X(s)+\dfrac{1}{s}\int_{-\infty}^{0^-} x(t)\,dt$
Time shifting	$x(t-t_0)u(t-t_0)$	$X(s)e^{-st_0} \quad t_0 \geq 0$
Frequency shifting	$x(t)e^{s_0t}$	$X(s-s_0)$
Frequency differentiation	$-tx(t)$	$\dfrac{dX(s)}{ds}$
Frequency integration	$\dfrac{x(t)}{t}$	$\int_{s}^{\infty} X(z)\,dz$
Scaling	$x(at), a \geq 0$	$\dfrac{1}{a}X\left(\dfrac{s}{a}\right)$
Time convolution	$x_1(t)*x_2(t)$	$X_1(s)X_2(s)$
Frequency convolution	$x_1(t)x_2(t)$	$\dfrac{1}{2\pi j}X_1(s)*X_2(s)$
Initial value	$x(0^+)$	$\lim_{s\to\infty} sX(s) \quad (n>m)$
Final value	$x(\infty)$	$\lim_{s\to 0} sX(s) \quad$ [poles of $sX(s)$ in LHP]

Example:

Find the time convolution y(t) = e^{2t} u(t)* e^{3t} u(t)

Solution:

$$y(t) = e^{2t}u(t) * e^{3t}u(t)$$

Taking the Laplace transform of both sides:

$$e^{\lambda t}u(t) \rightarrow \frac{1}{s-\lambda}$$

$$Y(s) = \frac{1}{s-2} \times \frac{1}{s-3}$$

$$Y(s) = \frac{1}{(s-2)(s-3)}$$

Using Partial Fractions:

$$Y(s) = \frac{k_1}{s-2} + \frac{k_2}{s-3}$$

Solution:

$$Y(s) = \frac{1}{(s-2)(s-3)} \qquad Y(s) = \frac{k_1}{s-2} + \frac{k_2}{s-3}$$

$$k_1 = \frac{1}{\cancel{(s-2)}(s-3)}\bigg|_{s=2} = \frac{1}{-1} = -1$$

$$k_2 = \frac{1}{(s-2)\cancel{(s-3)}}\bigg|_{s=3} = \frac{1}{1} = 1$$

$$Y(s) = -\frac{1}{s-2} + \frac{1}{s-3}$$

Taking the inverse Laplace transform: $e^{\lambda t}u(t) \rightarrow \frac{1}{s-\lambda}$

$$y(t) = -e^{2t}u(t) + e^{3t}u(t)$$

SOLUTION OF DIFFERENTIAL EQUATIONS

Example:

Solve the second-order linear differential equation:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

for the initial conditions $y(0^-)=2$, and $\dot{y}(0^-)=1$ and the input $x(t) = e^{-4t}u(t)$

Solution:

Taking the Laplace transform of the above differential equation:

$$[s^2Y(s) - sy(0^-) - \dot{y}(0^-)] + 5[sY(s) - y(0^-)] + 6Y(s) = [sX(s) - x(0^-)] + X(s)$$

Solution:

y(0⁻)=2, and $\dot{y}(0^-)$=1 and the input $x(t) = e^{-4t}u(t)$

x(0⁻)=0, and X(s) =1/(s+4)

$$[s^2Y(s) - sy(0^-) - \dot{y}(0^-)] + 5[sY(s) - y(0^-)] + 6Y(s) = [sX(s) - x(0^-)] + X(s)$$

$$[s^2Y(s) - 2s - 1] + 5[sY(s) - 2] + 6Y(s) = \left[\frac{s}{s+4} - 0\right] + \frac{1}{s+4}$$

$$s^2Y(s) + 5sY(s) + 6Y(s) = \frac{s+1}{s+4} + 2s + 11$$

$$Y(s)(s^2 + 5s + 6) = \frac{s+1}{s+4} + 2s + 11$$

Solution:

$$Y(s)(s^2+5s+6)=\frac{s+1}{s+4}+2s+11$$

$$Y(s)=\frac{\frac{s+1}{s+4}+2s+11}{s^2+5s+6}$$

$$Y(s)=\frac{s+1+(2s+11)(s+4)}{(s^2+5s+6)(s+4)}$$

$$Y(s)=\frac{s+1+2s^2+19s+44}{(s+2)(s+3)(s+4)}$$

$$Y(s)=\frac{2s^2+20s+45}{(s+2)(s+3)(s+4)}$$

Solution: Using Partial Fractions:

$$Y(s)=\frac{2s^2+20s+45}{(s+2)(s+3)(s+4)} \qquad Y(s)=\frac{k_1}{s+2}+\frac{k_2}{s+3}+\frac{k_3}{s+4}$$

$$k_1=\left.\frac{2s^2+20s+45}{\cancel{(s+2)}(s+3)(s+4)}\right|_{s=-2}=\frac{13}{2}=6.5$$

$$k_2=\left.\frac{2s^2+20s+45}{(s+2)\cancel{(s+3)}(s+4)}\right|_{s=-3}=\frac{3}{-1}=-3$$

$$k_3=\left.\frac{2s^2+20s+45}{(s+2)(s+3)\cancel{(s+4)}}\right|_{s=-4}=\frac{-3}{2}=-1.5$$

$$Y(s)=\frac{6.5}{s+2}-\frac{3}{s+3}-\frac{1.5}{s+4}$$

Solution:

$$Y(s) = \frac{6.5}{s+2} - \frac{3}{s+3} - \frac{1.5}{s+4}$$

Taking the inverse Laplace transform:

$$e^{\lambda t}u(t) \rightarrow \frac{1}{s-\lambda}$$

$$y(t) = 6.5e^{-2t}u(t) - 3e^{-3t}u(t) - 1.5e^{-4t}u(t)$$

$$y(t) = (6.5e^{-2t} - 3e^{-3t} - 1.5e^{-4t})u(t)$$

Zero Input response and Zero State Response

The Laplace transform method gives the total response, which includes zero-input response (ZIR) and zero-state response (ZSR) components.

The initial condition terms in the response give rise to the zero-input response.

Consider the previous example:

This term is due to the i/p signal

This term is due to initial conditions

$$Y(s) = \frac{\frac{s+1}{s+4}}{s^2+5s+6} + \frac{2s+11}{s^2+5s+6}$$

ZSR **ZIR**

Total Response

Zero Input response and Zero State Response

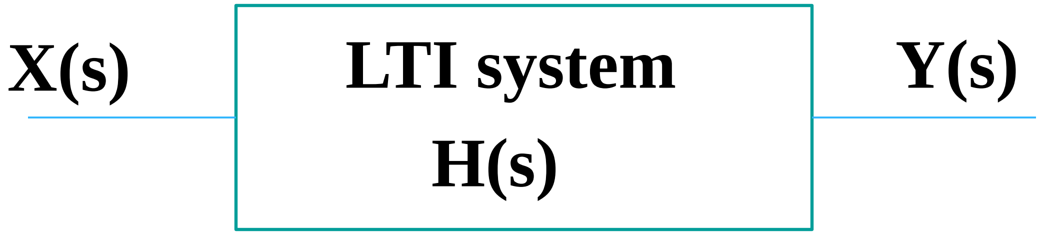
After taking the inverse Laplace transform of each term separately:

$$y(t) = (-0.5e^{-2t} + 2e^{-3t} - 1.5e^{-4t})u(t) + (7e^{-2t} - 5e^{-3t})u(t)$$

ZSR **ZIR**

Total Response

The Transfer Function H(s)



Consider a LTIC system, where the i/p signal is X(s), and the o/p signal is Y(s). The transfer function of the system is defined as:

$$H(s) = \frac{Y(s)}{X(s)}$$

Example:

Find the transfer function H(s) of an LTIC system described by the equation:

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = \frac{dx(t)}{dt} + x(t)$$

All the initial conditions are zero. Then find the zero-state response y(t) of the system if the input is $x(t) = 3e^{-5t}u(t)$

Solution:

Taking the Laplace transform of the above differential equation:

$$[s^2Y(s) - sy(0^-) - \dot{y}(0^-)] + 5[sY(s) - y(0^-)] + 6Y(s)$$
$$= [sX(s) - x(0^-)] + X(s)$$

All the initial conditions are zero.

Solution:

$$s^2Y(s) + 5sY(s) + 6Y(s) = sX(s) + X(s)$$

$$Y(s)(s^2 + 5s + 6) = X(s)(s + 1)$$

The transfer function H(s) is:

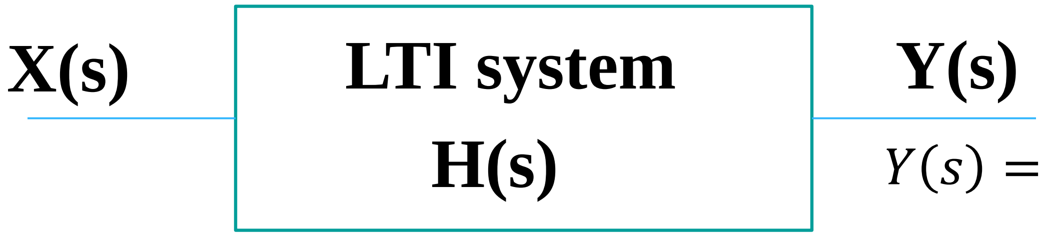
$$H(s) = \frac{Y(s)}{X(s)}$$

$$H(s) = \frac{s + 1}{s^2 + 5s + 6}$$

$$H(s) = \frac{s + 1}{(s + 2)(s + 3)}$$

Solution:

To get the system response:



$$Y(s) = X(s)H(s)$$

$$x(t) = 3e^{-5t}u(t) \qquad\qquad e^{\lambda t}u(t) \rightarrow \frac{1}{s-\lambda}$$

$$X(s) = \frac{3}{s+5}$$

$$Y(s) = X(s)H(s)$$

$$Y(s) = \frac{3(s+1)}{(s+2)(s+3)(s+5)}$$

Solution: Using Partial Fractions:

$$Y(s) = \frac{3(s+1)}{(s+2)(s+3)(s+5)} \qquad Y(s) = \frac{k_1}{s+2} + \frac{k_2}{s+3} + \frac{k_3}{s+5}$$

$$k_1 = \left.\frac{3(s+1)}{(s+3)(s+5)}\right|_{s=-2} = \frac{-3}{3} = -1$$

$$k_2 = \left.\frac{3(s+1)}{(s+2)(s+5)}\right|_{s=-3} = \frac{-6}{-2} = 3$$

$$k_3 = \left.\frac{3(s+1)}{(s+2)(s+3)}\right|_{s=-5} = \frac{-12}{6} = -2$$

$$Y(s) = -\frac{1}{s+2} + \frac{3}{s+3} - \frac{2}{s+5}$$

Solution:

$$Y(s) = -\frac{1}{s+2} + \frac{3}{s+3} - \frac{2}{s+5}$$

Taking the Inverse Laplace transform: $e^{\lambda t}u(t) \rightarrow \frac{1}{s-\lambda}$

$$y(t) = -e^{-2t}u(t) + 3e^{-3t}u(t) - 2e^{-5t}u(t)$$

$$y(t) = (3e^{-3t} - e^{-2t} - 2e^{-5t})u(t)$$

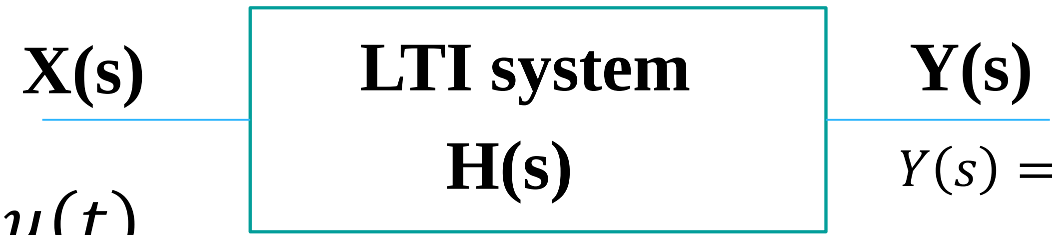
Example:

For an LTIC system with transfer function,

$$H(s) = \frac{s+5}{s^2+4s+3}$$

Find the system response y(t) to the input $x(t) = e^{-2t}u(t)$ if the system is initially in zero state.

Solution:



$$Y(s) = X(s)H(s)$$

$$x(t) = e^{-2t}u(t)$$

$$X(s) = \frac{1}{s+2}$$

Solution:

$$X(s) = \frac{1}{s+2} \qquad H(s) = \frac{s+5}{s^2+4s+3}$$

$$Y(s) = X(s)H(s)$$

$$Y(s) = \frac{(s+5)}{(s+2)(s^2+4s+3)}$$

$$Y(s) = \frac{(s+5)}{(s+2)(s+3)(s+1)}$$

Solution:

$$Y(s) = \frac{(s+5)}{(s+2)(s+3)(s+1)} \qquad Y(s) = \frac{k_1}{s+2} + \frac{k_2}{s+3} + \frac{k_3}{s+1}$$

$$k_1 = \frac{(s+5)}{(s+3)(s+1)}\Bigg|_{s=-2} = \frac{3}{-1} = -3$$

$$k_2 = \frac{(s+5)}{(s+2)(s+1)}\Bigg|_{s=-3} = \frac{2}{2} = 1$$

$$k_3 = \frac{(s+5)}{(s+2)(s+3)}\Bigg|_{s=-1} = \frac{4}{2} = 2$$

$$Y(s) = -\frac{3}{s+2} + \frac{1}{s+3} + \frac{2}{s+1}$$

Solution:

$$Y(s) = -\frac{3}{s+2} + \frac{1}{s+3} + \frac{2}{s+1}$$

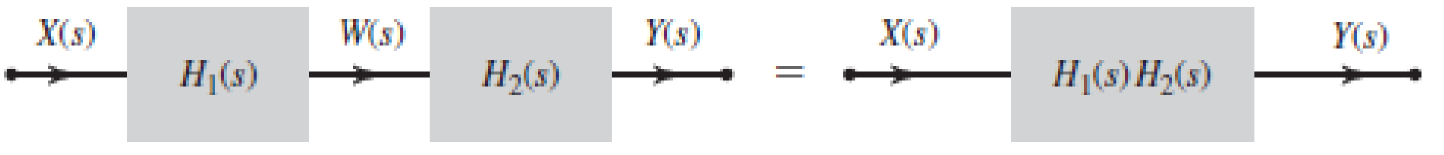
Taking the Inverse Laplace transform: $e^{\lambda t}u(t) \to \frac{1}{s-\lambda}$

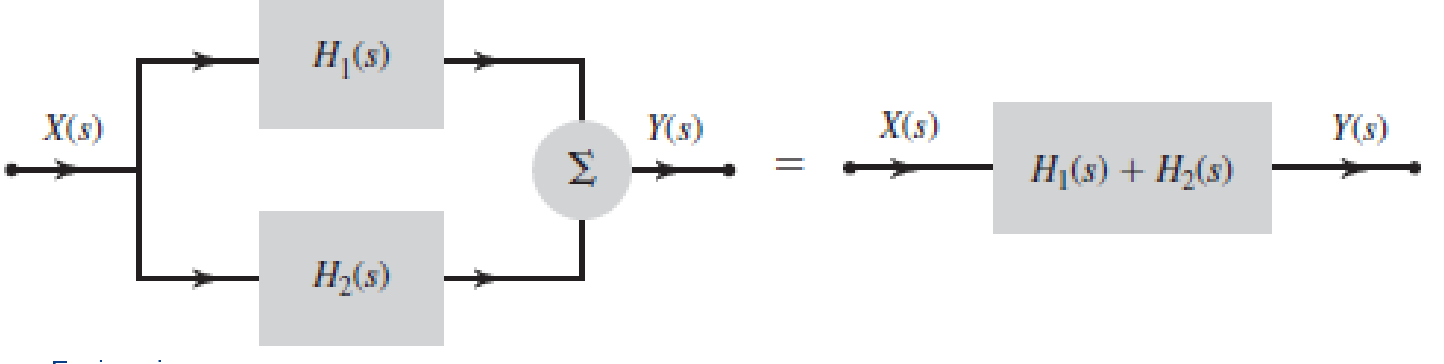
$$y(t) = -3e^{-2t}u(t) + e^{-3t}u(t) + 2e^{-t}u(t)$$

$$y(t) = (2e^{-t} - 3e^{-2t} + e^{-3t})u(t)$$

Block Diagrams

Large systems may consist of an enormous number of components or elements. In such cases, it is convenient to represent a system by suitably interconnected subsystems, each of which can be readily analyzed. Each subsystem can be characterized in terms of its input–output relationships. A linear system can be characterized by its transfer function H(s).





Example:

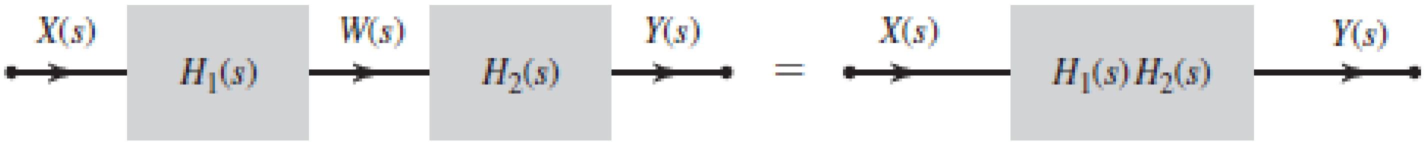
Consider two LTIC systems. The first has transfer function $H_1(s) = \frac{2s}{s+1}$ and the second has a transfer function $H_2(s) = \frac{s+3}{s+5}$.

1. Determine the overall impulse response if the two systems are connected in series (cascaded).

2. Determine the overall impulse response if the two systems are connected in parallel.

Solution:

1. Block diagrams are cascaded:



$$H_1(s) = \frac{2s}{s+1} \qquad H_2(s) = \frac{s+3}{s+5}$$

$$H(s) = H_1(s)H_2(s)$$

$$H(s) = \frac{2s}{s+1} \times \frac{s+3}{s+5}$$

$$H(s) = \frac{2s(s+3)}{(s+1)(s+5)}$$

To get the overall impulse response, we should get the inverse Laplace transform of H(s).

Solution:

Using partial Fraction Method: the numerator order should be less than the denominator order

$$H(s) = \frac{2s(s+3)}{(s+1)(s+5)} \qquad H(s) = 2 + \frac{k_1}{s+1} + \frac{k_2}{s+5}$$

$$k_1 = \left.\frac{2s(s+3)}{(s+5)}\right|_{s=-1} = \frac{-4}{4} = -1$$

$$k_2 = \left.\frac{2s(s+3)}{(s+1)}\right|_{s=-5} = \frac{20}{-4} = -5$$

$$H(s) = 2 - \frac{1}{s+1} - \frac{5}{s+5}$$

Solution:

$$H(s) = 2 - \frac{1}{s+1} - \frac{5}{s+5}$$

Taking the Inverse Laplace transform:

$$\delta(t) \rightarrow 1$$

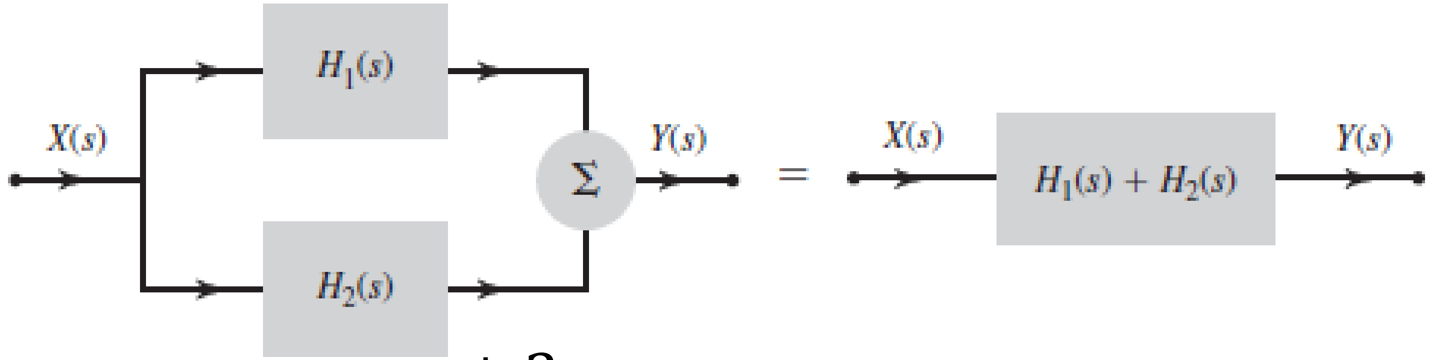
$$e^{\lambda t}u(t) \rightarrow \frac{1}{s-\lambda}$$

$$h(t) = 2\delta(t) - e^{-t}u(t) - 5e^{-5t}u(t)$$

Solution:

2. Block diagrams are Parallel:

$$H_1(s) = \frac{2s}{s+1}$$



$$H_2(s) = \frac{s+3}{s+5}$$

$$H(s) = H_1(s) + H_2(s)$$

$$H(s) = \frac{2s}{s+1} + \frac{s+3}{s+5}$$

To get the overall impulse response, we should get the inverse Laplace transform of H(s).

Solution:

$$H(s) = \frac{2s}{s+1} + \frac{s+3}{s+5}$$

$$H(s) = \frac{2s}{s+1} + \frac{s}{s+5} + \frac{3}{s+5}$$

$$H(s) = 2s\left(\frac{1}{s+1}\right) + s\left(\frac{1}{s+5}\right) + \frac{3}{s+5}$$

Taking the Inverse Laplace transform:

$$\frac{dx(t)}{dt} \rightleftharpoons sX(s) - x(0^-)$$

$$e^{\lambda t}u(t) \to \frac{1}{s-\lambda}$$

$$e^{-5t}u(t) \to \frac{1}{s+5}$$

$$\frac{d}{dt}\left(e^{-5t}u(t)\right) \to \frac{s}{s+5}$$

$$-5e^{-5t}u(t) + \delta(t) \to \frac{s}{s+5}$$

$$e^{-t}u(t) \to \frac{1}{s+1}$$

$$\frac{d}{dt}\left(e^{-t}u(t)\right) \to \frac{s}{s+1}$$

$$-e^{-t}u(t) + \delta(t) \to \frac{s}{s+1}$$

Solution:

$$H(s) = 2s\left(\frac{1}{s+1}\right) + s\left(\frac{1}{s+5}\right) + \frac{3}{s+5}$$

$$h(t) = 2\big(\delta(t) - e^{-t}\,u(t)\big) + \left(\delta(t) - 5e^{-5t}\,u(t)\right) + 3e^{-5t}u(t)$$

$$h(t) = 3\delta(t) - 2e^{-t}\,u(t) - 2e^{-5t}\,u(t)$$