Chapter 3

CONTINUOUS-TIME SIGNAL ANALYSIS: THE FOURIER SERIES AND FOURIER TRANSFORM

Introduction:

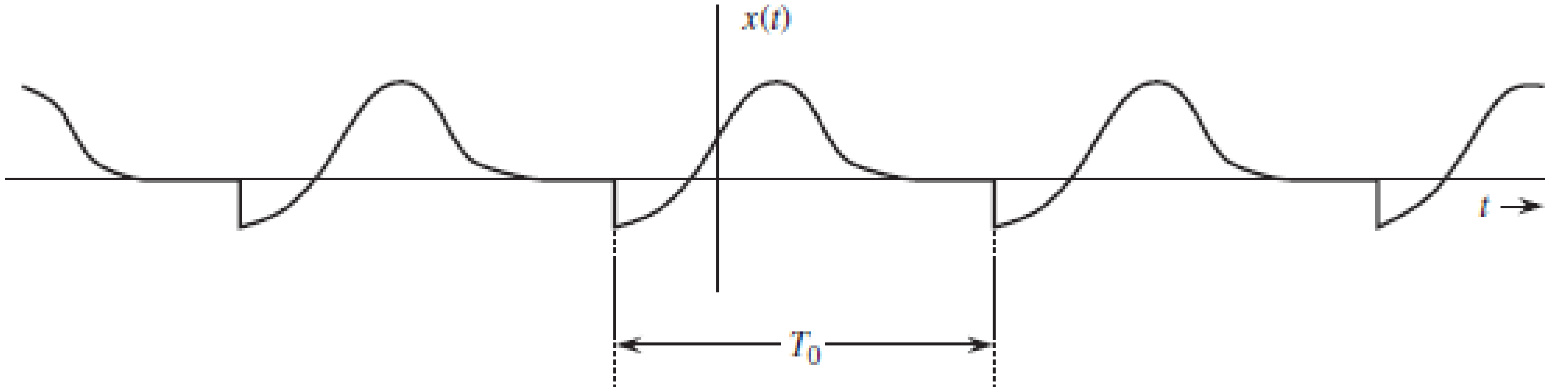
Electrical engineers instinctively think of signals in terms of their frequency spectra and think of systems in terms of their frequency response.

Most teenagers know about the audible portion of audio signals having a bandwidth of about 20 kHz and the need for good-quality speakers to respond up to 20 kHz. This is basically thinking in the frequency domain.

In this chapter we show that a periodic signal can be represented as a sum of sinusoids (or exponentials) of various frequencies (called Fourier Series). These results are extended to aperiodic signals using Fourier transform.

PERIODIC SIGNAL REPRESENTATION BY TRIGONOMETRIC FOURIER SERIES

A periodic signal x(t) of a period T_0 could be expressed in terms of a trigonometric Fourier series as follows:



$$x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

where $\omega_0 = \frac{2\pi}{T_0}$ is called the fundamental frequency.

PERIODIC SIGNAL REPRESENTATION BY TRIGONOMETRIC FOURIER SERIES

$$x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + b_n \sin n\omega_0 t$$

Where $a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$ $\quad a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos n\omega_0 t\, dt$

$$b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\sin n\omega_0 t\, dt$$

So, A periodic signal x(t) with a period T_0 can be expressed as a sum of a sinusoid of frequency f_0 ($f_0 = 1/T_0$) and all its harmonics.

Compact Form of Fourier Series

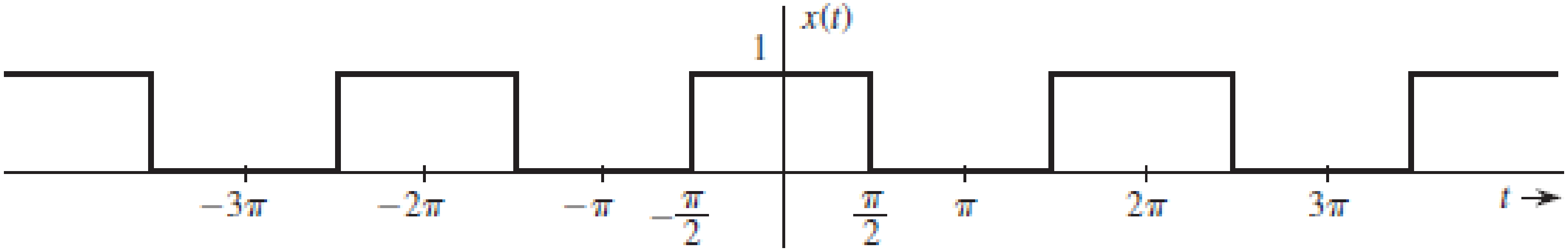
When x(t) is real, coefficients a_n and b_n are real for all n, and the trigonometric Fourier series can be expressed in a compact form:

$$x(t) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

where $C_0 = a_0$ $\quad C_n = \sqrt{a_n^2 + b_n^2}$ $\quad \theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$

Example:

Find the compact trigonometric Fourier series for the square-pulse periodic signal shown in figure and sketch its Fourier spectrum.



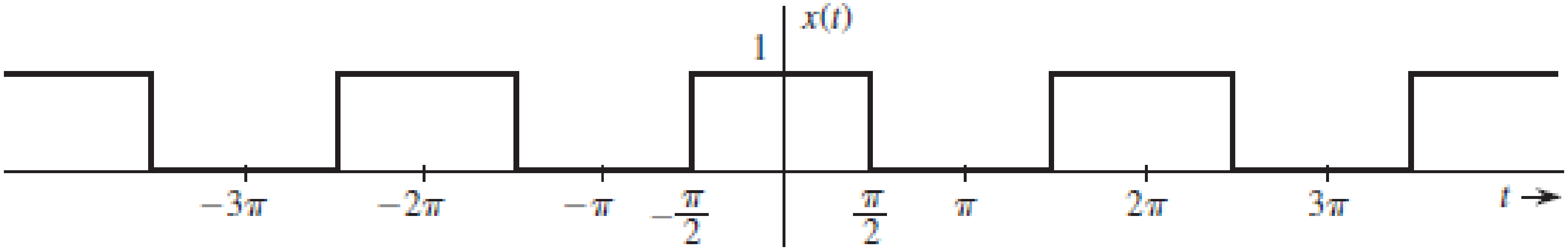
Solution:

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

$$a_0 = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1dt$$

$$a_0 = \frac{1}{2\pi} \times \pi = 0.5$$

Solution:



$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos n\omega_0 t\, dt \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1$$

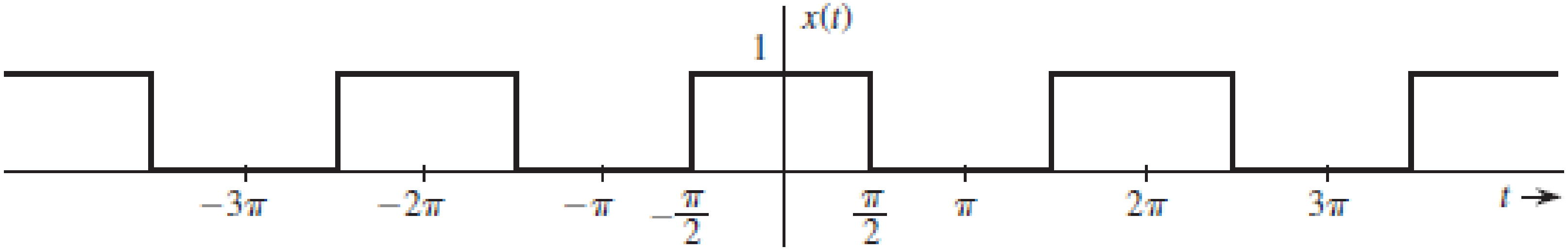
$$a_n = \frac{2}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \times \cos nt\, dt$$

$$a_n = \frac{1}{\pi}\left[\frac{\sin nt}{n}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$a_n = \frac{1}{\pi}\left(\frac{\sin n\frac{\pi}{2} - \sin -n\frac{\pi}{2}}{n}\right)$$

$$a_n = \frac{2}{\pi}\left(\frac{\sin n\frac{\pi}{2}}{n}\right)$$

Solution:



$$b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\sin n\omega_0 t\,dt \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1$$

$$b_n = \frac{2}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \times \sin nt\,dt$$

$$b_n = 0$$

Solution:

$$C_0 = a_0 = 0.5$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

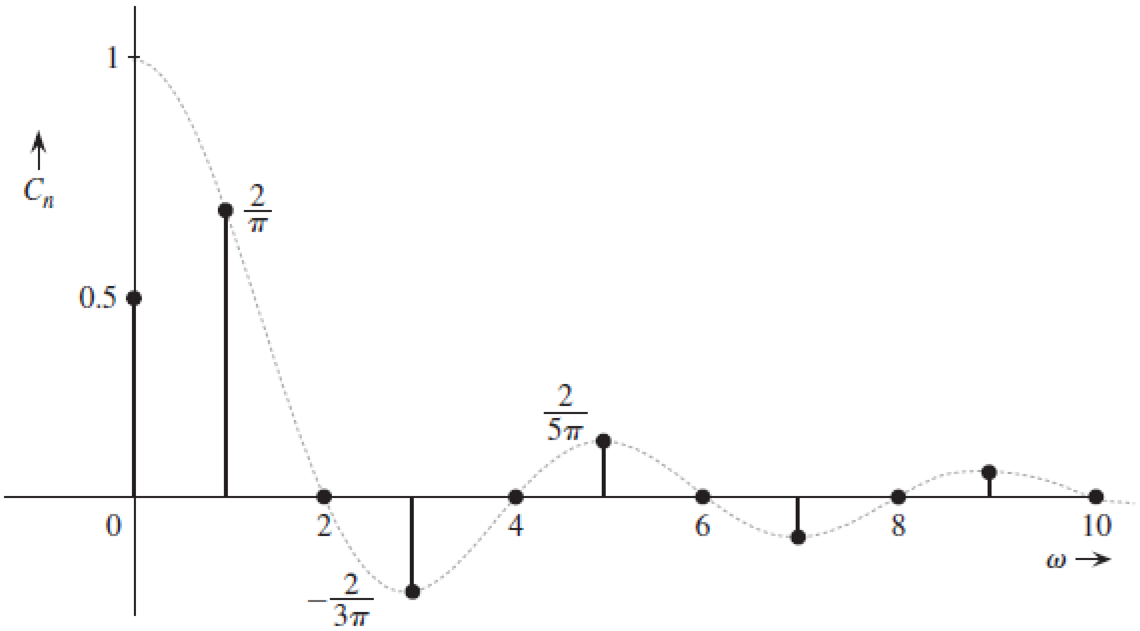
$$C_n = \frac{2}{\pi}\left(\frac{\sin n\frac{\pi}{2}}{n}\right)$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = 0$$

$$x(t) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

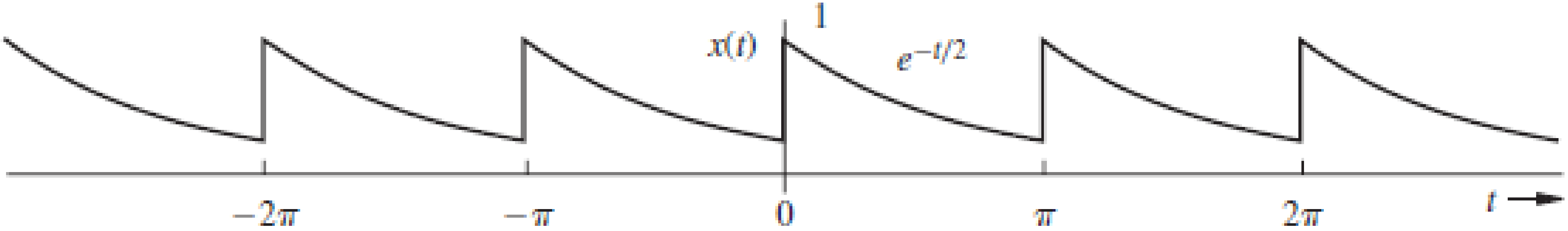
$$x(t) = 0.5 + \sum_{n=1}^{\infty} \frac{2}{\pi}\left(\frac{\sin n\frac{\pi}{2}}{n}\right)\cos(nt)$$

Solution:



Example:

Find the compact trigonometric Fourier series for the periodic signal x(t) shown in figure. Sketch the amplitude spectrum for x(t).



$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$	$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}(b\sin bx + a\cos bx)}{a^2+b^2}$

Solution:

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt \qquad\qquad a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}dt$$

Solution:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-t/2} dt$$

$$a_0 = \frac{1}{\pi}\left[\frac{e^{-t/2}}{-0.5}\right]_0^{\pi}$$

$$a_0 = \frac{1}{\pi}\left(\frac{e^{-\pi/2} - e^{0}}{-0.5}\right) = 0.504$$

$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos n\omega_0 t\, dt \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} = 2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\cos 2nt\, dt$$

Solution:

$$a_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\cos 2nt\, dt$$

$$a_n = \frac{2}{\pi}\left[e^{-t/2}\left(\frac{2\text{nsin}\, 2nt - 0.5\cos 2nt}{(-0.5)^2 + (2n)^2}\right)\right]_0^{\pi}$$

$$a_n = \frac{2}{\pi}\left[e^{-\pi/2}\left(\frac{2\text{nsin}\, 2n\pi - 0.5\cos 2n\pi}{(-0.5)^2 + (2n)^2}\right) - e^{0}\left(\frac{0 - 0.5}{(-0.5)^2 + (2n)^2}\right)\right]$$

$$a_n = \frac{2}{\pi}\left[e^{-\pi/2}\left(\frac{-0.5}{(-0.5)^2 + (2n)^2}\right) - \left(\frac{0 - 0.5}{(-0.5)^2 + (2n)^2}\right)\right]$$

$$a_n = \frac{2}{\pi}\left[e^{-\pi/2}\left(\frac{-0.5}{0.25 + 4n^2}\right) - \left(\frac{-0.5}{0.25 + 4n^2}\right)\right]$$

Solution:

$$a_n = \frac{2}{\pi}\left[e^{-\pi/2}\left(\frac{-0.5}{0.25+4n^2}\right) - \left(\frac{-0.5}{0.25+4n^2}\right)\right]$$

$$a_n = \frac{2}{\pi}\left(\frac{-0.5}{0.25+4n^2}\right)\left[e^{-\pi/2} - 1\right]$$

$$a_n = \left(\frac{0.25}{0.25+4n^2}\right)$$

$$b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\sin n\omega_0 t\, dt \qquad \omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} = 2$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\sin 2nt\, dt$$

Solution:

$$b_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\sin 2nt\, dt$$

$$b_n = \frac{2}{\pi}\left[e^{-t/2}\left(\frac{-0.5\sin 2nt - 2n\cos 2nt}{(-0.5)^2 + (2n)^2}\right)\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left|e^{-\pi/2}\left(\frac{-0.5\sin 2n\pi - 2n\cos 2n\pi}{(-0.5)^2 + (2n)^2}\right) - e^0\left(\frac{0 - 2n}{(-0.5)^2 + (2n)^2}\right)\right|$$

$$b_n = \frac{2}{\pi}\left|e^{-\pi/2}\left(\frac{-2n}{0.25 + 4n^2}\right) - \left(\frac{-2n}{0.25 + 4n^2}\right)\right|$$

$$b_n = \frac{2}{\pi}\left(\frac{-2n}{0.25 + 4n^2}\right)\left[e^{-\pi/2} - 1\right]$$

$$b_n = \left(\frac{n}{0.25 + 4n^2}\right)$$

Solution:

$$C_0 = a_0 = 0.504$$

$$C_n = \sqrt{a_n^2 + b_n^2}$$

$$C_n = \sqrt{\left(\frac{0.25}{0.25 + 4n^2}\right)^2 + \left(\frac{n}{0.25 + 4n^2}\right)^2}$$

$$C_n = \frac{1}{0.25 + 4n^2}\sqrt{(0.25)^2 + n^2}$$

$$C_n = \frac{0.5}{0.25 + 4n^2}\sqrt{4(0.25)^2 + 4n^2}$$

$$C_n = \frac{0.5}{0.25 + 4n^2}\sqrt{0.25 + 4n^2}$$

$$C_n = \frac{0.5}{\sqrt{0.25 + 4n^2}}$$

Solution:

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right)$$

$$a_n = \left(\frac{0.25}{0.25 + 4n^2}\right) \qquad b_n = \left(\frac{n}{0.25 + 4n^2}\right)$$

$$\theta_n = \tan^{-1}(-4n)$$

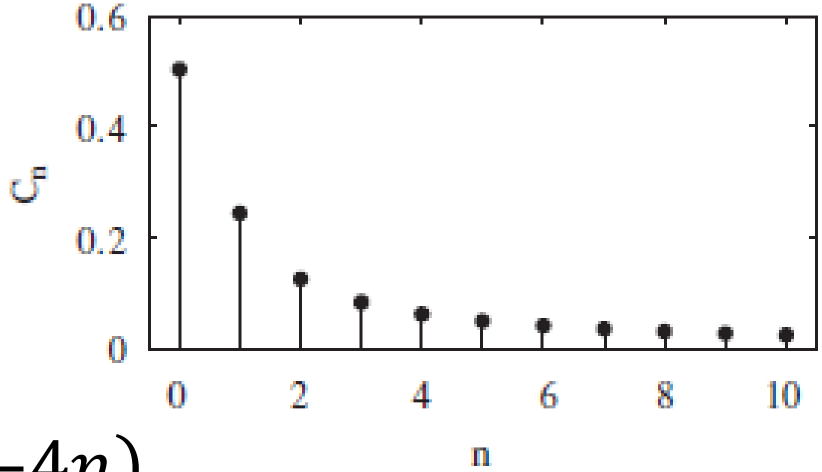
$$x(t) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$x(t) = 0.504 + \sum_{n=1}^{\infty} \frac{0.25}{\sqrt{0.25 + 4n^2}} \cos(2nt - \tan^{-1}(4n))$$

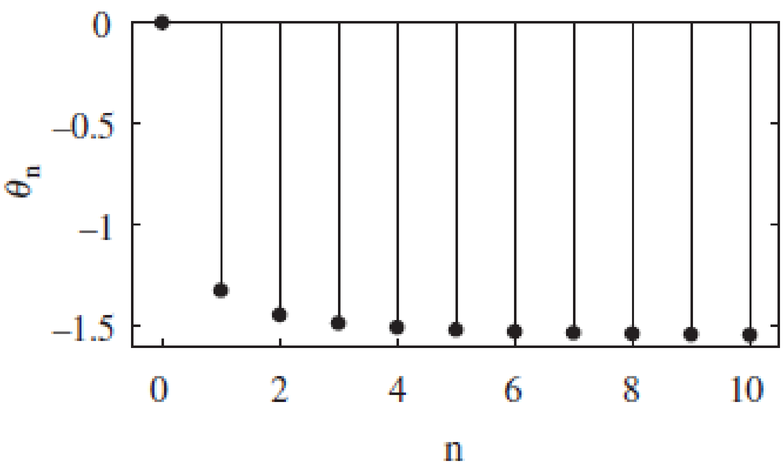
Solution:

$$C_0 = 0.504$$

$$C_n = \frac{0.5}{\sqrt{0.25 + 4n^2}}$$

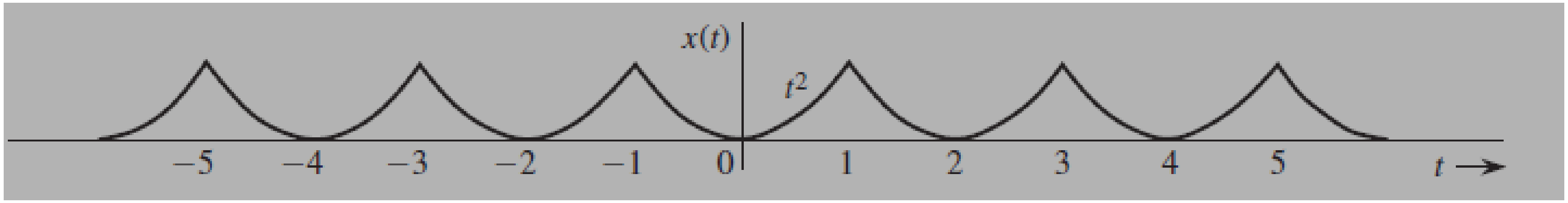


$$\theta_n = \tan^{-1}(-4n)$$



Example:

Find the compact trigonometric Fourier series for the periodic signal x(t) shown in figure. Sketch the amplitude spectrum for x(t).



$$\int x^2 \cos ax\,dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2-2}{a^3}\sin ax$$

Solution:

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} t^2 dt$$

$$a_0 = \frac{1}{2}\left[\frac{t^3}{3}\right]_{-1}^{1}$$

$$a_0 = \frac{1}{2}\left[\frac{1-(-1)}{3}\right] = \frac{1}{3}$$

Solution:

$$a_n = \frac{2}{T_0}\int_{-T_o/2}^{T_o/2} x(t)\cos n\omega_0 t\, dt \qquad \omega_0 = \frac{2\pi}{2} = \pi$$

$$a_n = \int_{-1}^{1} t^2 \cos n\pi t\, dt \qquad a_n = 2\int_{0}^{1} t^2 \cos n\pi t\, dt$$

$$\int x^2 \cos ax dx = \frac{2x\cos ax}{a^2} + \frac{a^2x^2 - 2}{a^3}\sin ax$$

$$a_n = 2\left[\frac{2t\cos n\pi t}{n^2\pi^2} + \frac{n^2\pi^2t^2 - 2}{n^3\pi^3}\sin n\pi t\right]_0^1$$

$$a_n = 2\left(\frac{2\cos n\pi}{n^2\pi^2}\right)$$

Solution:

$$b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\sin n\omega_0 t\, dt \qquad \omega_0 = \frac{2\pi}{2} = \pi$$

$$b_n = \int_{-1}^{1} t^2 \sin n\pi t\, dt \qquad b_n = 0$$

$$x(t) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

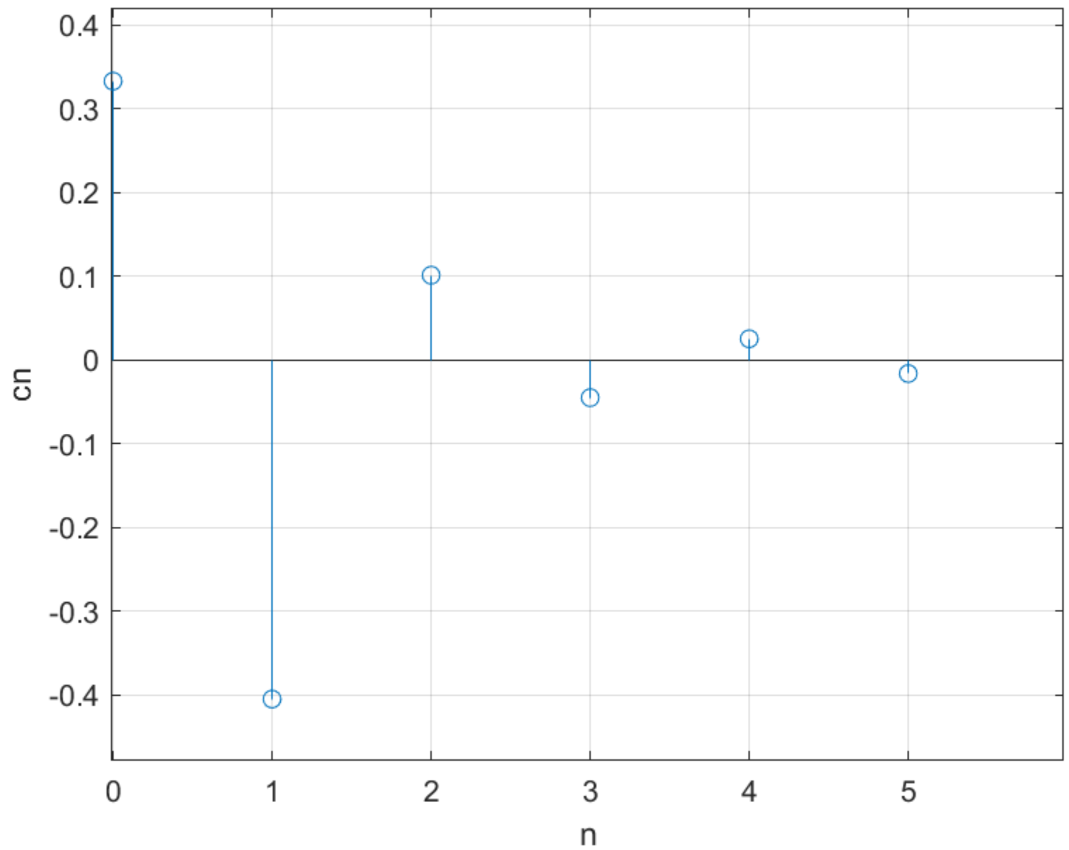
$$C_0 = a_0 = \frac{1}{3} \qquad C_n = \sqrt{a_n^2 + b_n^2} \qquad C_n = \frac{4\cos n\pi}{n^2\pi^2}$$

$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = 0$$

Solution:

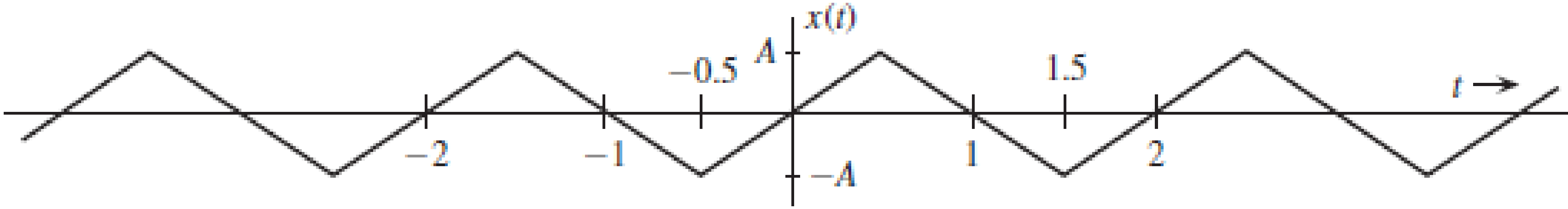
$$x(t) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$x(t) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4\cos n\pi}{n^2\pi^2} \cos(n\pi t)$$



Example:

Find the compact trigonometric Fourier series for the periodic signal x(t) shown in figure. Sketch the amplitude spectrum for x(t).



$$x(t) = \begin{cases} 2At & |t| < \frac{1}{2} \\ 2A(1-t) & \frac{1}{2} < t < \frac{3}{2} \end{cases} \qquad \int x \sin ax\, dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$$

Solution:

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)dt$$

$$a_0 = \frac{1}{2}\int_{-1}^{1} x(t)dt$$

$$a_0 = 0$$

Solution:

$$a_n = \frac{2}{T_0}\int_{-T_o/2}^{T_o/2} x(t)\cos n\omega_0 t\, dt$$

$$\omega_0 = \frac{2\pi}{2} = \pi$$

$$a_n = \int_{-1}^{1} x(t)\cos n\pi t\, dt$$

$$a_n = 0$$

$$b_n = \frac{2}{T_0}\int_{0}^{T_0} x(t)\sin n\omega_0 t\, dt$$

$$b_n = \int_{-1}^{1} x(t)\sin n\pi t\, dt$$

$$b_n = 2\int_{0}^{1} x(t)\sin n\pi t\, dt$$

Solution:

$$b_n = 2\int_0^1 x(t)\sin n\pi t\,dt$$

$$x(t) = \begin{cases} 2At & |t| < \frac{1}{2} \\ 2A(1-t) & \frac{1}{2} < t < \frac{3}{2} \end{cases}$$

$$b_n = 2\int_0^{0.5} 2At\sin n\pi t\,dt + 2\int_{0.5}^1 2A(1-t)\sin n\pi t\,dt$$

$$b_n = 4A\int_0^{0.5} t\sin n\pi t\,dt - 4A\int_{0.5}^1 t\sin n\pi t\,dt + 4A\int_{0.5}^1 \sin n\pi t\,dt$$

$$\int x\sin ax\,dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$$

Solution:

$$\int x \sin ax\,dx = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$$

$$b_n = 4A\int_0^{0.5} t\sin n\pi t\,dt - 4A\int_{0.5}^{1} t\sin n\pi t\,dt + 4A\int_{0.5}^{1} \sin n\pi t\,dt$$

$$b_n = 4A\left[-\frac{t\cos n\pi t}{n\pi} + \frac{\sin n\pi t}{n^2\pi^2}\right]_0^{0.5} - 4A\left[-\frac{t\cos n\pi t}{n\pi} + \frac{\sin n\pi t}{n^2\pi^2}\right]_{0.5}^{1} + 4A\left[-\frac{\cos n\pi t}{n\pi}\right]_{0.5}^{1}$$

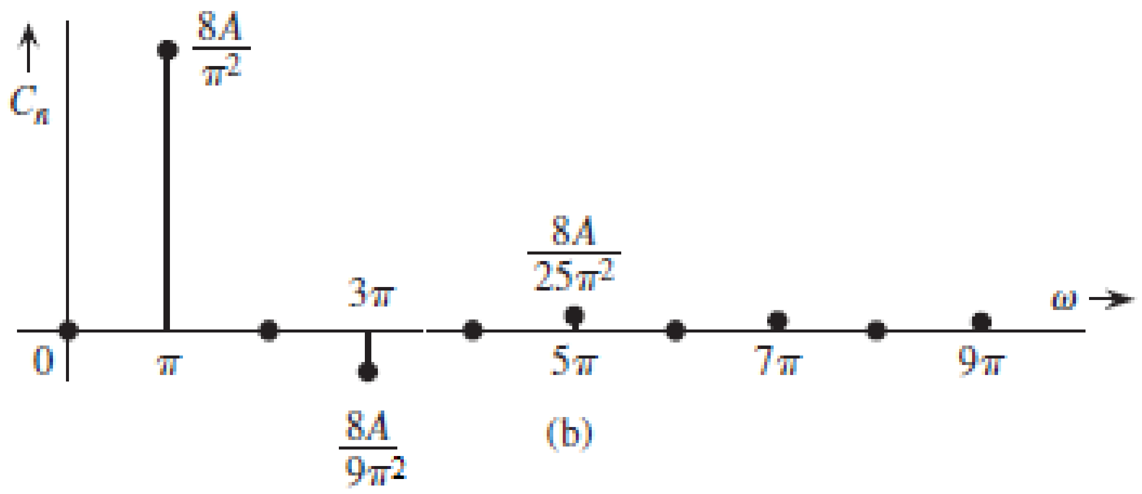
$$b_n = 4A\left(-\frac{0.5\cos\frac{n\pi}{2}}{n\pi} + \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right) - 4A\left(-\frac{\cos n\pi}{n\pi} + \frac{\sin n\pi}{n^2\pi^2} + \frac{0.5\cos\frac{n\pi}{2}}{n\pi} - \frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right) + 4A\left(-\frac{\cos n\pi}{n\pi} + \frac{\cos\frac{n\pi}{2}}{n\pi}\right)$$

Solution: $b_n = 8A\left(\frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right)$

$$x(t) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

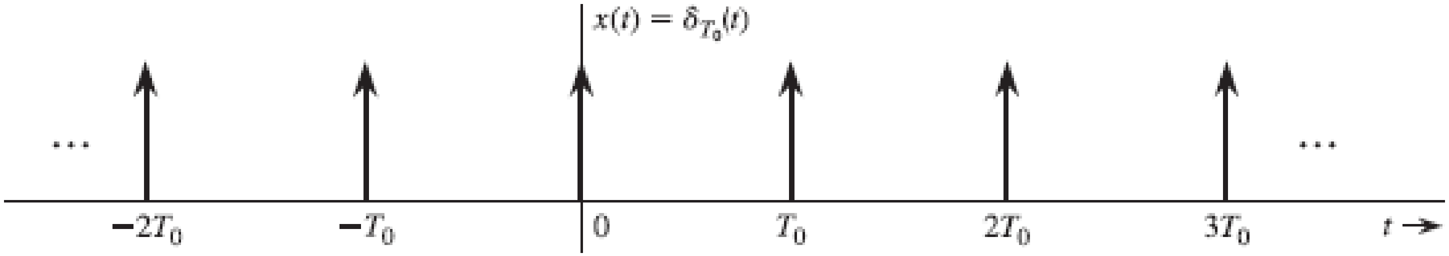
$C_0 = a_0 = 0$ $\qquad C_n = \sqrt{a_n^2 + b_n^2}$ $\qquad C_n = 8A\left(\frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right)$

$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = -\frac{\pi}{2}$ $\qquad x(t) = \sum_{n=1}^{\infty} 8A\left(\frac{\sin\frac{n\pi}{2}}{n^2\pi^2}\right)\cos\left(n\omega_0 t - \frac{\pi}{2}\right)$



Example:

Find the compact trigonometric Fourier series for the periodic for the impulse train shown in figure. Sketch the amplitude spectrum for x(t).



Solution:

$$a_0 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)dt$$

$$a_0 = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t)dt$$

$$a_0 = \frac{1}{T_0}$$

Solution:

$$a_n = \frac{2}{T_0}\int_{-T_o/2}^{T_o/2} x(t)\cos n\omega_0 t\, dt$$

$$a_n = \frac{2}{T_0}\int_{-T_o/2}^{T_o/2} \delta(t)\cos n\omega_0 t\, dt$$

$$\boldsymbol{\phi(t)\delta(t) = \phi(0)\delta(t)}$$

$$a_n = \frac{2}{T_0}\cos 0 \int_{-T_o/2}^{T_o/2} \delta(t) dt$$

$$\boldsymbol{\int_{-\infty}^{\infty} \delta(t) dt = 1}$$

$$a_n = \frac{2}{T_0}$$

Solution:

$$b_n = \frac{2}{T_0}\int_{-T_o/2}^{T_o/2} x(t)\sin n\omega_0 t\, dt$$

$$b_n = \frac{2}{T_0}\int_{-T_o/2}^{T_o/2} \delta(t)\sin n\omega_0 t\, dt \qquad \boldsymbol{\phi(t)\delta(t) = \phi(0)\delta(t)}$$

$$b_n = \frac{2}{T_0}\sin 0\int_{-T_o/2}^{T_o/2} \delta(t) dt$$

$$b_n = 0$$

Solution:

$$C_0 = a_0 = \frac{1}{T_0} \qquad C_n = \sqrt{a_n^2 + b_n^2} \qquad C_n = \frac{2}{T_0}$$

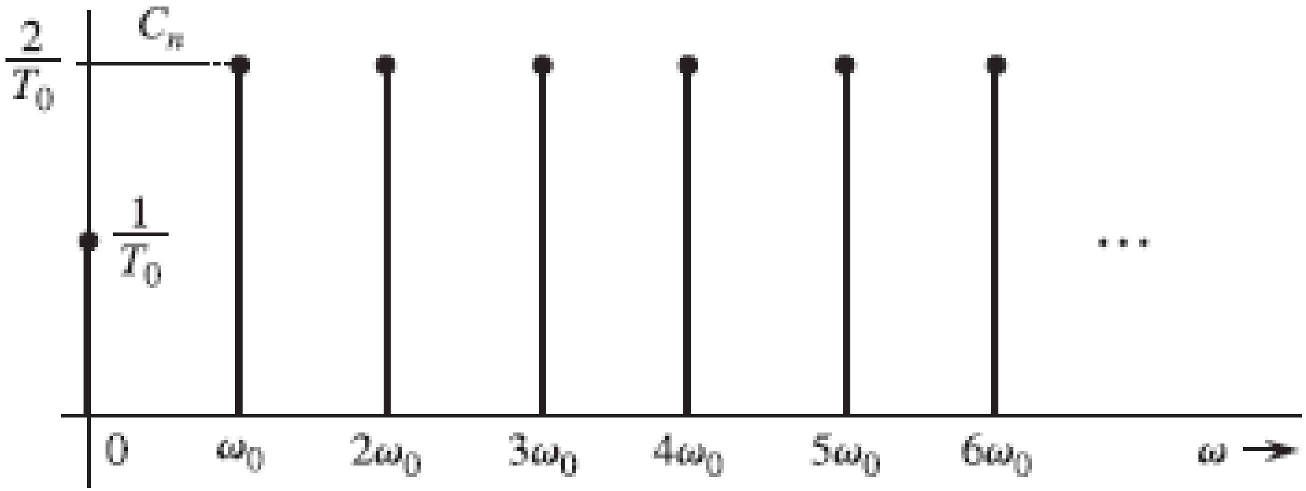
$$\theta_n = \tan^{-1}\left(\frac{-b_n}{a_n}\right) = 0$$

$$x(t) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

$$x(t) = \frac{1}{T_0} + \sum_{n=1}^{\infty} \frac{2}{T_0} \cos(n\omega_0 t)$$

Solution:

$$C_0 = \frac{1}{T_0} \qquad C_n = \frac{2}{T_0}$$



Parseval's Theorem

Parseval's theorem states that the power of a periodic signal is equal to the sum of the powers of its Fourier components.

$$x(t) = C_o + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \theta_n)$$

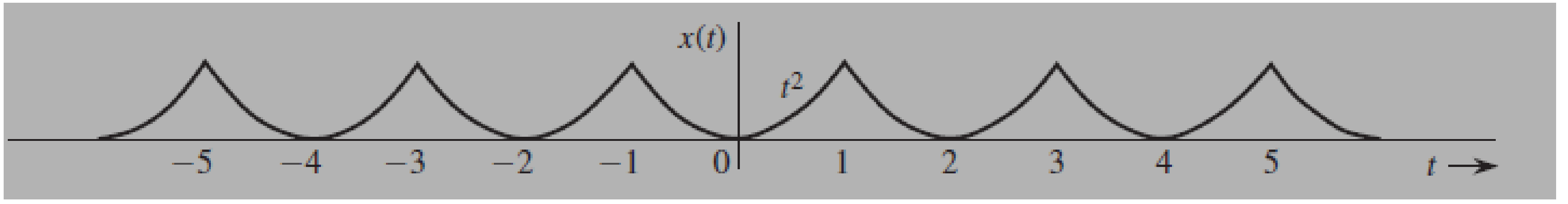
The power of the periodic signal x(is) is given by:

$$P_x = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2$$

Example:

Verify the Parseval's theorem for the signal given in slide 19, given that:

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$$



Solution:

$$P_x=\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2dt$$

$$P_x=\frac{1}{2}\int_{-1}^{1}t^4dt$$

$$P_x=\frac{1}{2}\left[\frac{t^5}{5}\right]_{-1}^{1}$$

$$P_x=\frac{1}{2}\left(\frac{1^5}{5}-\frac{(-1)^5}{5}\right)$$

$$P_x=0.2$$

Solution:

Using the Fourier series representation of x(t) in slide

$$x(t) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4\cos n\pi}{n^2\pi^2}\cos(n\pi t)$$

$$P_x = C_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} C_n^2$$

$$P_x = \left(\frac{1}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{4\cos n\pi}{n^2\pi^2}\right)^2$$

$$P_x = \left(\frac{1}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16\cos^2(n\pi)}{n^4\pi^4}$$

$$P_x = \frac{1}{9} + \frac{8}{\pi^4}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

$$P_x = \frac{1}{9} + \frac{8}{\pi^4} \times \frac{\pi^4}{90}$$

$$P_x = 0.2$$

THE FOURIER TRANSFORM

We succeeded in representing periodic signals as a sum of (everlasting) sinusoids. The Fourier integral developed in this section extends this spectral representation to aperiodic signals.

The **Fourier transform** X(f) of aperiodic signal x(t) is given by:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

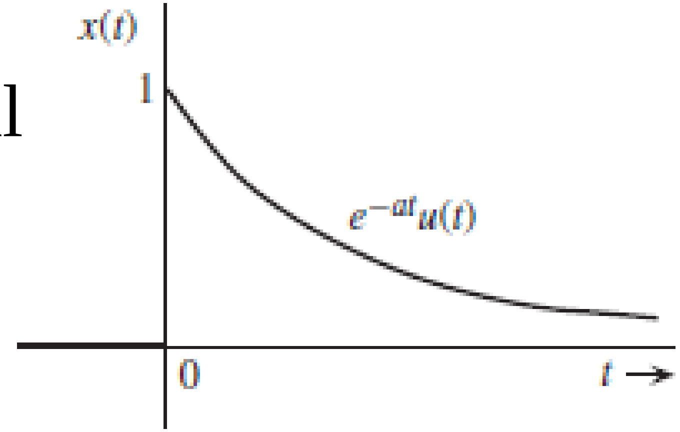
The **Inverse Fourier transform** to get x(t) from X(f) is given by:

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

Example:

Find the Fourier transform of the signal

$$x(t) = e^{-at}u(t)$$



Solution:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$X(f) = \int_{0}^{\infty} e^{-at}e^{-j2\pi ft}dt$$

$$X(f) = \int_{0}^{\infty} e^{-(a+j2\pi f)t}dt$$

$$X(f) = \left[\frac{e^{-(a+j2\pi f)t}}{-(a+j2\pi f)}\right]_0^{\infty}$$

$$X(f) = \left(\frac{e^{-\infty} - e^{0}}{-(a+j2\pi f)}\right)$$

$$X(f) = \frac{1}{a+j2\pi f}$$

Solution:

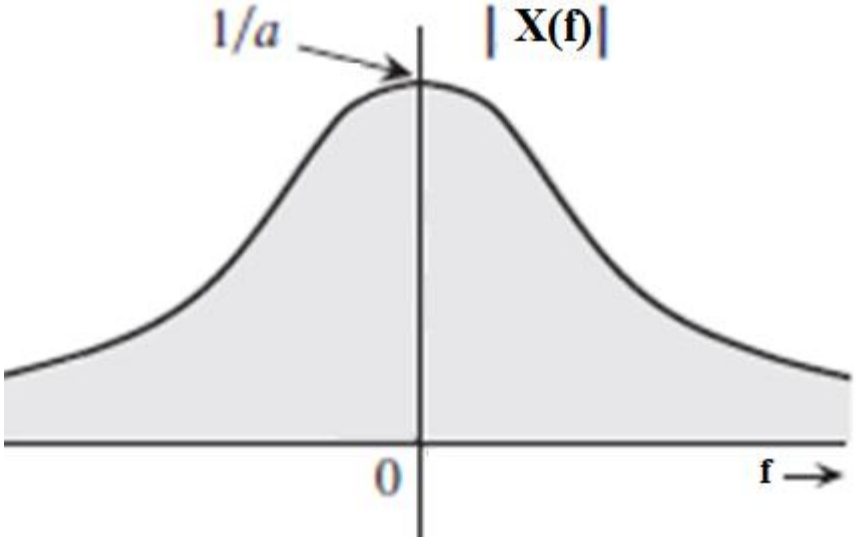
$$X(f) = \frac{1}{a + j2\pi f}$$

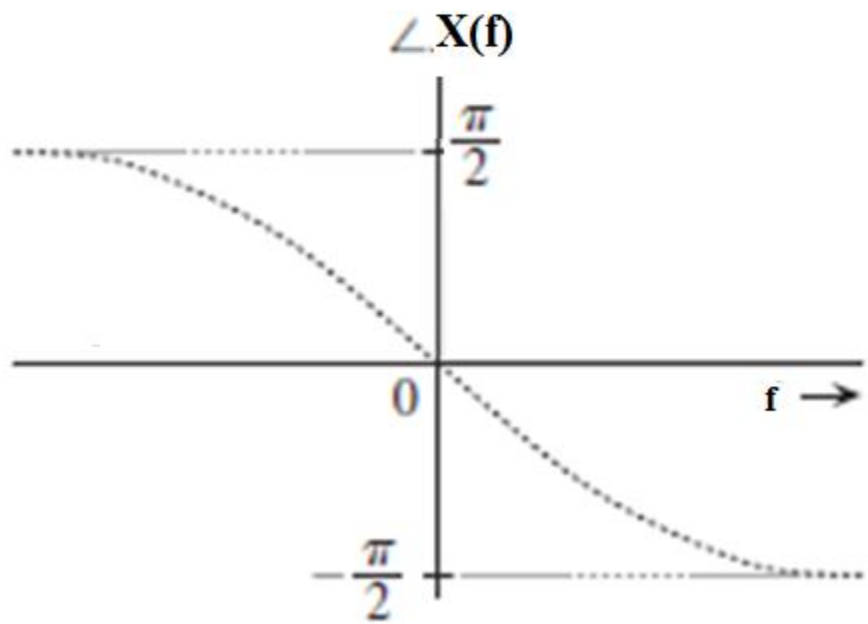
The amplitude spectrum is:

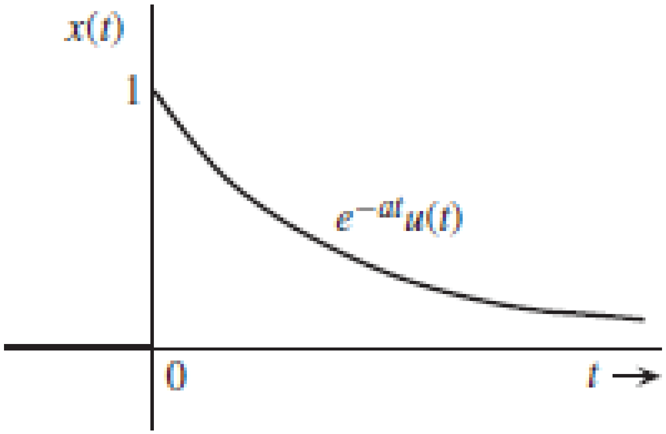
$$|X(f)| = \frac{1}{\sqrt{a^2 + (2\pi f)^2}}$$

The phase spectrum is:

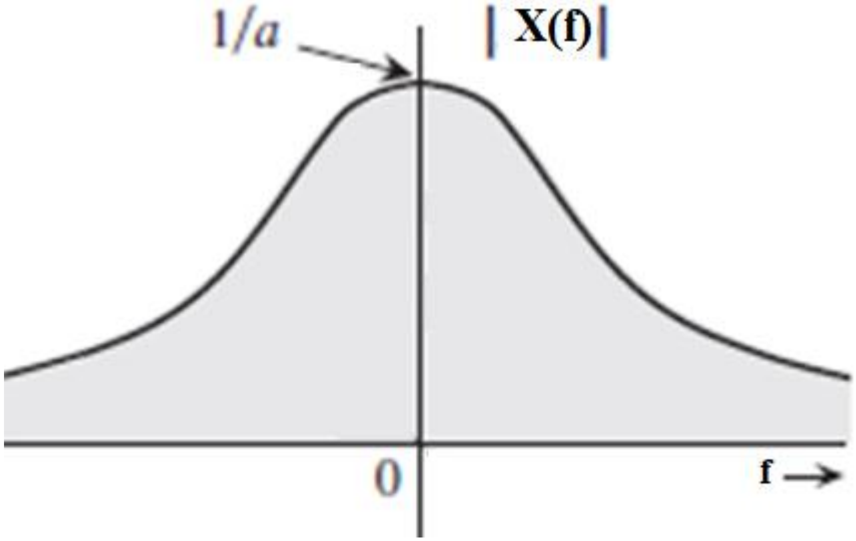
$$\angle X(\omega) = -\tan^{-1}\left(\frac{2\pi f}{a}\right)$$







$\rightleftharpoons$



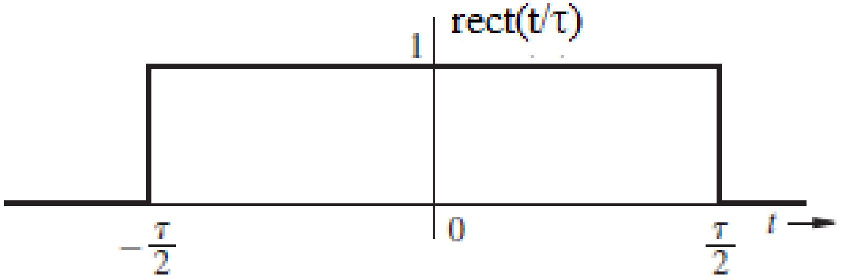
$$e^{-at}u(t) \quad \rightleftharpoons \quad \frac{1}{a + j2\pi f}$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

1. Rectangular Function:

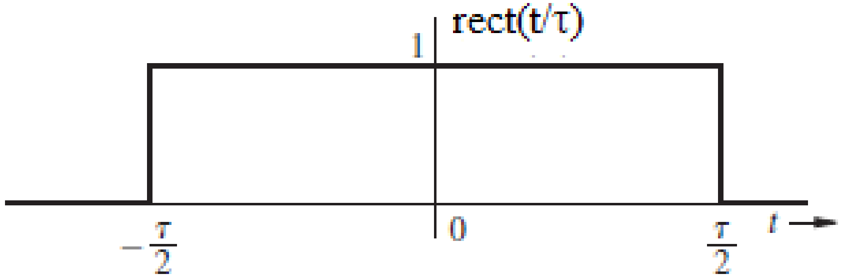
Find the Fourier transform of the signal $x(t) = rect\left(\frac{t}{\tau}\right)$

$$rect\left(\frac{t}{\tau}\right) = \begin{cases} 1 & |t| < \frac{\tau}{2} \\ 0 & |t| > \frac{\tau}{2} \end{cases}$$



TRANSFORMS OF SOME USEFUL FUNCTIONS

1. Rectangular Function:



$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

$$X(f) = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{-j2\pi f t} dt$$

$$X(f) = \left[\frac{e^{-j2\pi f t}}{-j2\pi f}\right]_{-\frac{\tau}{2}}^{\frac{\tau}{2}}$$

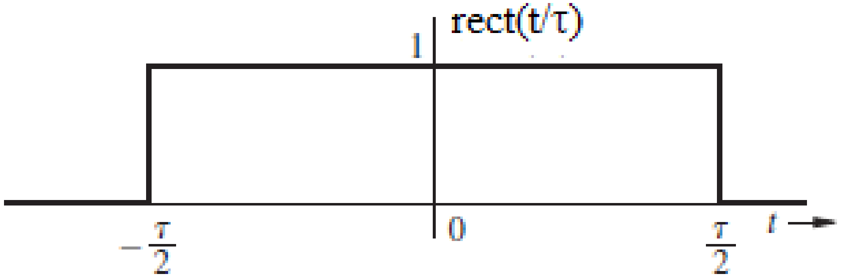
$$X(f) = \frac{e^{-j2\pi f\frac{\tau}{2}} - e^{j2\pi f\frac{\tau}{2}}}{-j2\pi f}$$

$$X(f) = \frac{1}{\pi f}\left(\frac{e^{j\pi f\tau} - e^{-j\pi f\tau}}{2j}\right)$$

$$X(f) = \frac{1}{\pi f}\sin(\pi f\tau)$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

1. Rectangular Function:

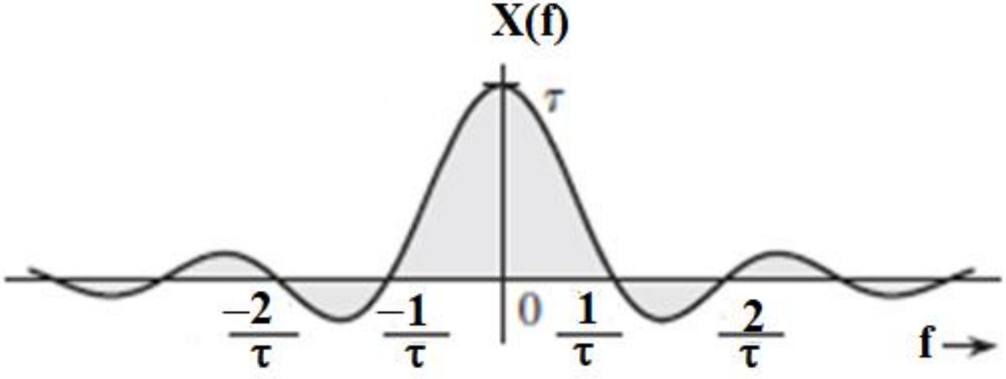


$$X(f) = \frac{1}{\pi f}\sin(\pi f \tau)$$

$$X(f) = \frac{\tau}{\pi f \tau}\sin(\pi f \tau)$$

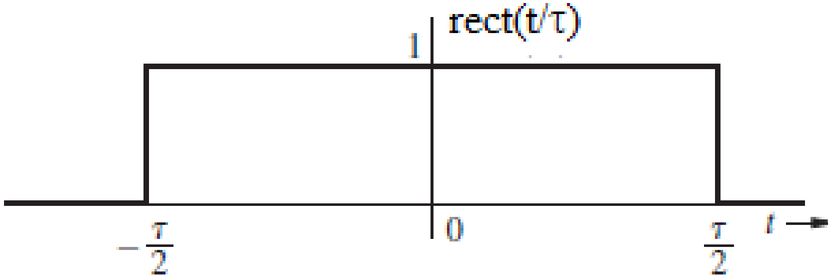
$$X(f) = \tau\frac{\sin(\pi f \tau)}{\pi f \tau}$$

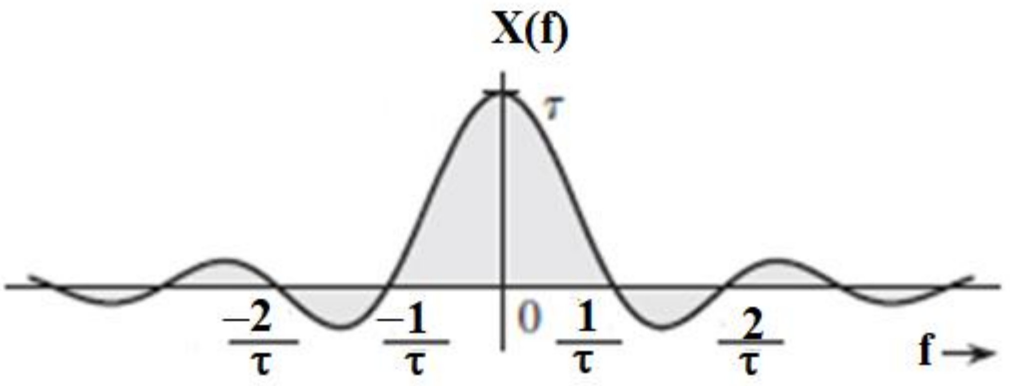
$$X(f) = \tau\, Sa(\pi f \tau)$$



TRANSFORMS OF SOME USEFUL FUNCTIONS

1. Rectangular Function:





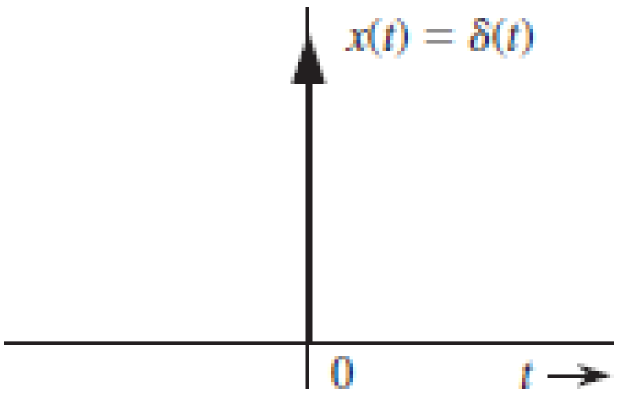
$$rect\left(\frac{t}{\tau}\right) \rightleftharpoons \tau\, Sa(\pi f \tau)$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

2. Unit impulse function

Find the Fourier transform of the signal $x(t) = \delta(t)$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$



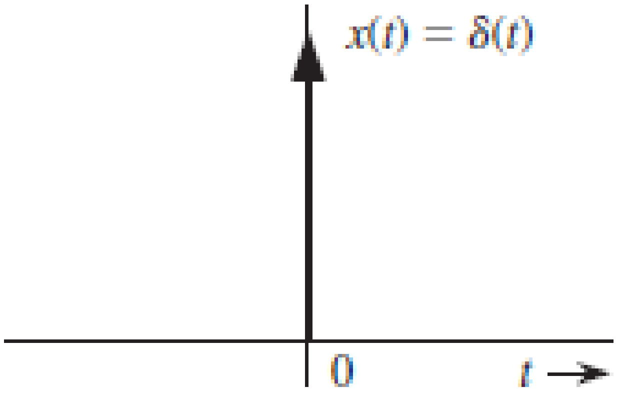
$$X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt$$

$$X(f) = \int_{-\infty}^{\infty} \delta(t) dt$$

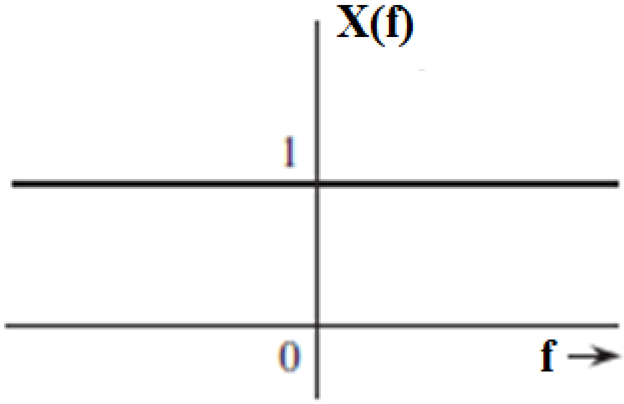
$$X(f) = 1$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

2. Unit impulse function



$\rightleftharpoons$



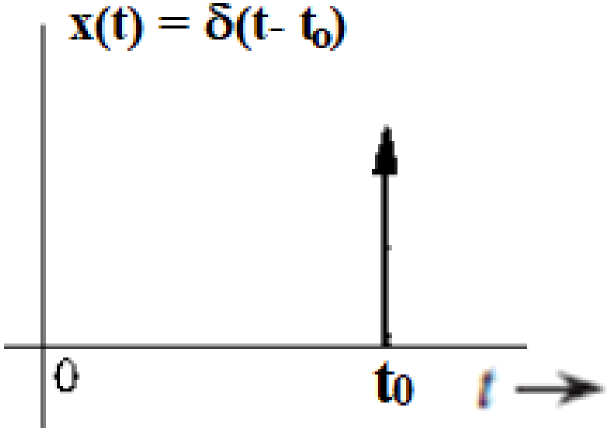
$$\delta(t) \qquad \rightleftharpoons \qquad 1$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

Shifted Unit impulse function

Find the Fourier transform of the signal $x(t) = \delta(t - t_o)$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$



$$X(f) = \int_{-\infty}^{\infty} \delta(t - t_o)e^{-j2\pi ft}dt$$

$$X(f) = e^{-j2\pi ft_o} \int_{-\infty}^{\infty} \delta(t - t_o)dt$$

$$X(f) = e^{-j2\pi ft_o}$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

Shifted Unit impulse function

$$\delta(t) \quad \rightleftharpoons \quad 1$$

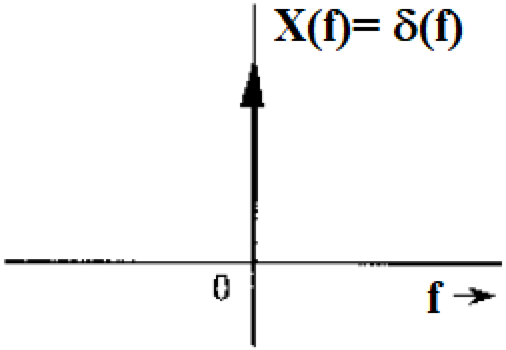
$$\delta(t - t_o) \quad \rightleftharpoons \quad 1\, e^{-j2\pi f t_o}$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

Unit impulse function in frequency domain

Find the Inverse Fourier transform of the signal $X(f) = \delta(f)$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$



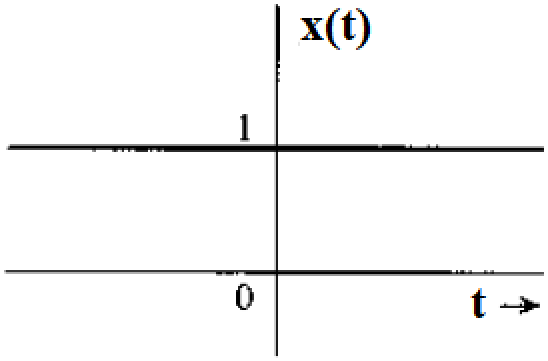
$$x(t) = \int_{-\infty}^{\infty} \delta(f) e^{j2\pi f t} dt$$

$$x(t) = \int_{-\infty}^{\infty} \delta(f) dt$$

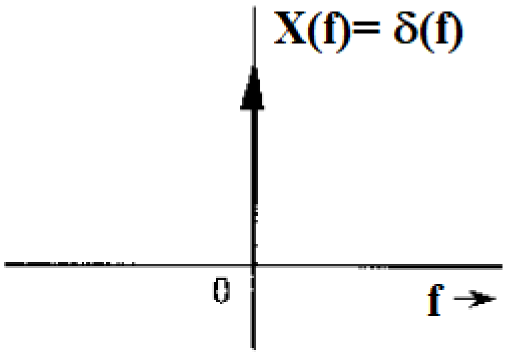
$$x(t) = 1$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

Unit impulse function in frequency domain



$\rightleftharpoons$



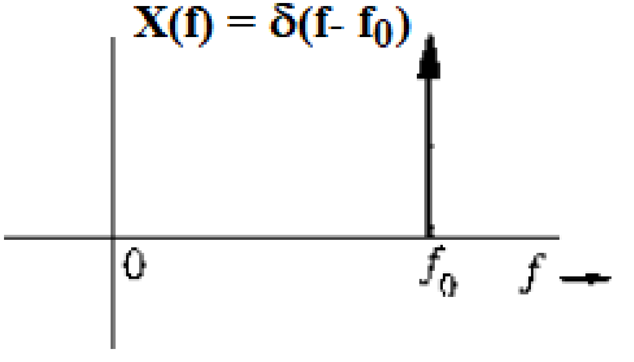
$$1 \quad \rightleftharpoons \quad \delta(f)$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

Shifted Unit impulse function in frequency domain

Find the Inverse Fourier transform of the signal $X(f) = \delta(f - f_0)$

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$



$$x(t) = \int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi f t} dt$$

$$x(t) = e^{j2\pi f_0 t} \int_{-\infty}^{\infty} \delta(f - f_0) dt$$

$$x(t) = e^{j2\pi f_0 t}$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

Shifted Unit impulse function in frequency domain

$$1 \rightleftharpoons \delta(f)$$

$$1\, e^{j2\pi f_0 t} \rightleftharpoons \delta(f - f_0)$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

3. Sinusoidal Functions $cos(2\pi f_0 t)$ and $\sin(2\pi f_0 t)$

Find the Fourier transform of the signal $x(t) = Acos(2\pi f_0 t)$

$$Acos(2\pi f_0 t) = \frac{A}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

$$1 \rightleftharpoons \delta(f)$$

$$\frac{A}{2} \rightleftharpoons \frac{A}{2}\delta(f)$$

$$\frac{A}{2}\ e^{j2\pi f_0 t} \rightleftharpoons \frac{A}{2}\delta(f - f_0)$$

$$\frac{A}{2}\ e^{-j2\pi f_0 t} \rightleftharpoons \frac{A}{2}\delta(f + f_0)$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

$$A\cos(2\pi f_0 t) = \frac{A}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

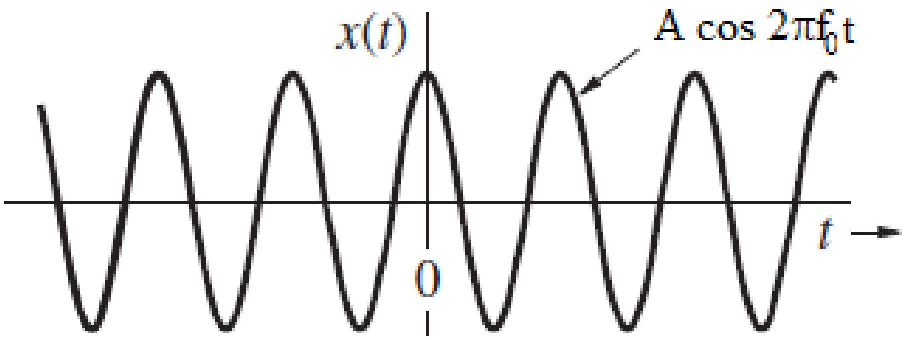
$$\frac{A}{2}\ e^{j2\pi f_0 t} \rightleftharpoons \frac{A}{2}\delta(f - f_0)$$

$$+$$

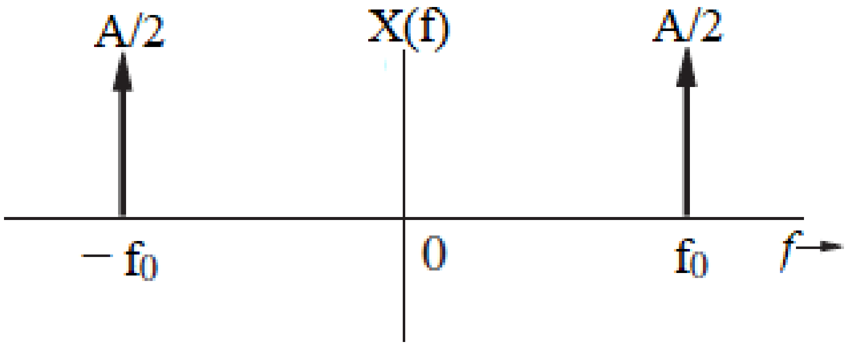
$$\frac{A}{2}\ e^{-j2\pi f_0 t} \rightleftharpoons \frac{A}{2}\delta(f + f_0)$$

$$A\cos(2\pi f_0 t) \rightleftharpoons \frac{A}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right)$$

TRANSFORMS OF SOME USEFUL FUNCTIONS



$\rightleftharpoons$



$$A\cos(2\pi f_0 t) \quad \rightleftharpoons \quad \frac{A}{2}\left(\delta(f + f_0) + \delta(f - f_0)\right)$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

3. Sinusoidal Functions $cos(2\pi f_0 t)$ and $\sin(2\pi f_0 t)$

Find the Fourier transform of the signal $x(t) = Asin(2\pi f_0 t)$

$$Asin(2\pi f_0 t) = \frac{A}{2j}\left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right)$$

$$1 \rightleftharpoons \delta(f)$$

$$\frac{A}{2} \rightleftharpoons \frac{A}{2}\delta(f)$$

$$\frac{A}{2j}\ e^{j2\pi f_0 t} \rightleftharpoons \frac{A}{2j}\delta(f - f_0)$$

$$\frac{A}{2j}\ e^{-j2\pi f_0 t} \rightleftharpoons \frac{A}{2j}\delta(f + f_0)$$

TRANSFORMS OF SOME USEFUL FUNCTIONS

$$A\sin(2\pi f_0 t) = \frac{A}{2j}\left(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}\right)$$

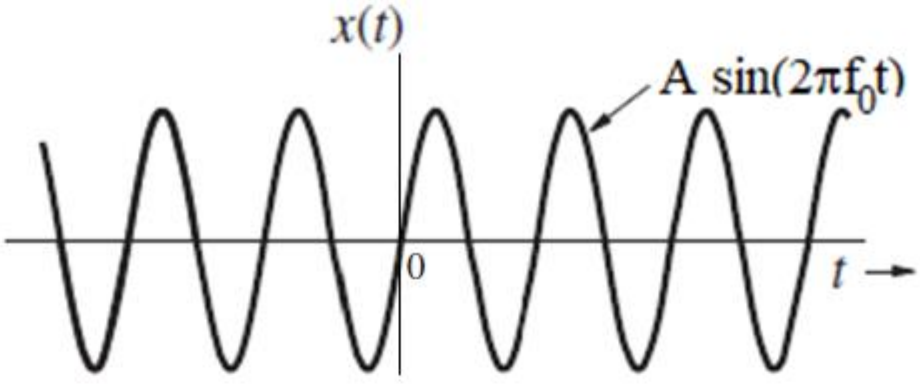
$$\frac{A}{2j}\, e^{j2\pi f_0 t} \rightleftharpoons \frac{A}{2j}\delta(f - f_0)$$

$$-$$

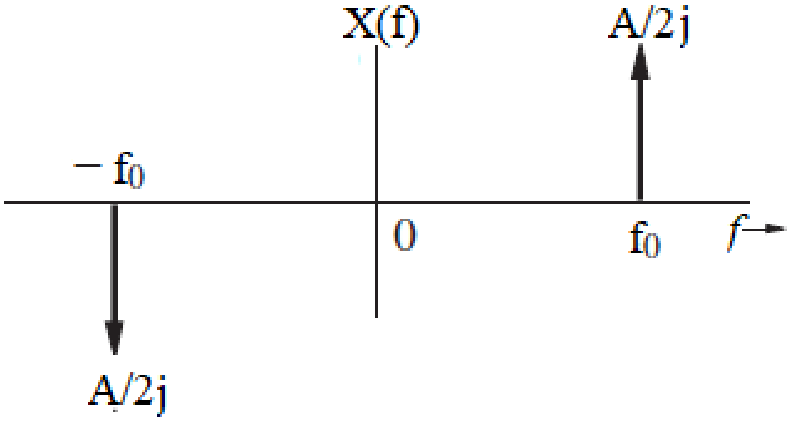
$$\frac{A}{2j}\, e^{-j2\pi f_0 t} \rightleftharpoons \frac{A}{2j}\delta(f + f_0)$$

$$A\sin(2\pi f_0 t) \rightleftharpoons \frac{A}{2j}\left(\delta(f - f_0) - \delta(f + f_0)\right)$$

TRANSFORMS OF SOME USEFUL FUNCTIONS



$\rightleftharpoons$



$$A \sin(2\pi f_0 t) \quad \rightleftharpoons \quad \frac{A}{2j}\left(\delta(f - f_0) - \delta(f + f_0)\right)$$

SOME PROPERTIES OF THE FOURIER TRANSFORM

1. LINEARITY:

The Fourier transform is linear that is if:

$$x_1(t) \rightleftharpoons X_1(f) \quad \text{and} \quad x_2(t) \rightleftharpoons X_2(f)$$

Then, for all constants a_1 and a_2, we have:

$$\boxed{a_1 x_1(t) + a_2 x_2(t) \rightleftharpoons a_1 X_1(f) + a_2 X_2(f)}$$

SOME PROPERTIES OF THE FOURIER TRANSFORM

2. CONJUGATION AND CONJUGATE SYMMETRY

The conjugation property is,

$$x(t) \rightleftharpoons X(f) \quad \text{Then} \quad x^*(t) \rightleftharpoons X^*(-f)$$

The conjugate symmetry property states that if x(t) is real, then:

$$X(-f) = X^*(f)$$

SOME PROPERTIES OF THE FOURIER TRANSFORM

3. DUALITY:

The duality property states that if

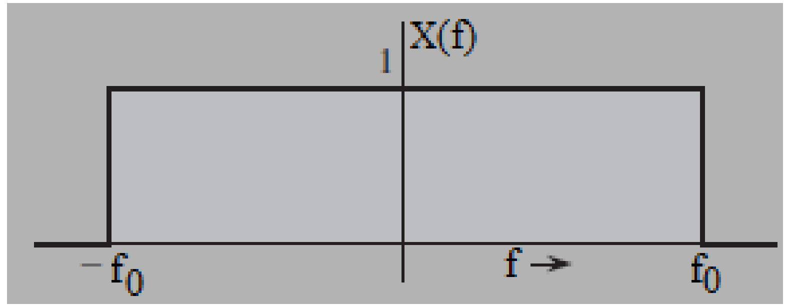
$$x(t) \rightleftharpoons X(f)$$

Then

$$X(t) \rightleftharpoons x(-f)$$

Example:

Find the inverse Fourier transform of X(f) illustrated in figure.

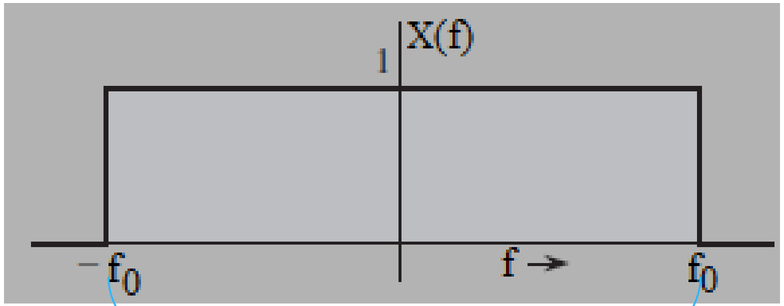


Solution:

$$rect\left(\frac{t}{\tau}\right) \rightleftharpoons \tau\, Sa(\pi f \tau)$$

$$\tau\, Sa(\pi t \tau) \rightleftharpoons rect\left(\frac{-f}{\tau}\right)$$

Solution:



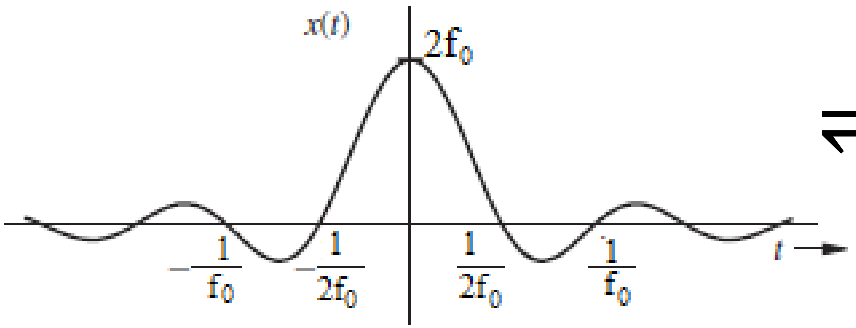
But in figure the width is $2f_0$ not τ

$$\tau\, Sa(\pi t\tau) \rightleftharpoons rect\left(\frac{f}{\tau}\right)$$

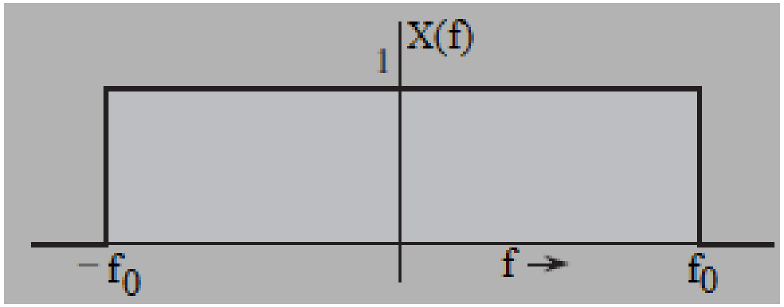
$\tau = 2f_0$

$$2f_0 Sa(2\pi f_0 t) \rightleftharpoons rect\left(\frac{f}{2f_0}\right)$$

Solution:



$\rightleftharpoons$



$$2f_0 Sa(2\pi f_0 t) \quad \rightleftharpoons \quad rect\left(\frac{f}{2f_0}\right)$$

Example:

Find the Fourier transform of the signal $x(t) = \frac{1}{a+j2\pi t}$

Solution:

$$x(t) \rightleftharpoons X(f)$$

$$X(t) \rightleftharpoons x(-f)$$

$$e^{-at}u(t) \rightleftharpoons \frac{1}{a+j2\pi f}$$

$$\frac{1}{a+j2\pi t} \rightleftharpoons e^{af}u(-f)$$

SOME PROPERTIES OF THE FOURIER TRANSFORM

4. TIME SCALING:

The time scaling property states that if

$$x(t) \rightleftharpoons X(f)$$

Then, for any real constant a

$$x(at) \rightleftharpoons \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Special case: a = –1

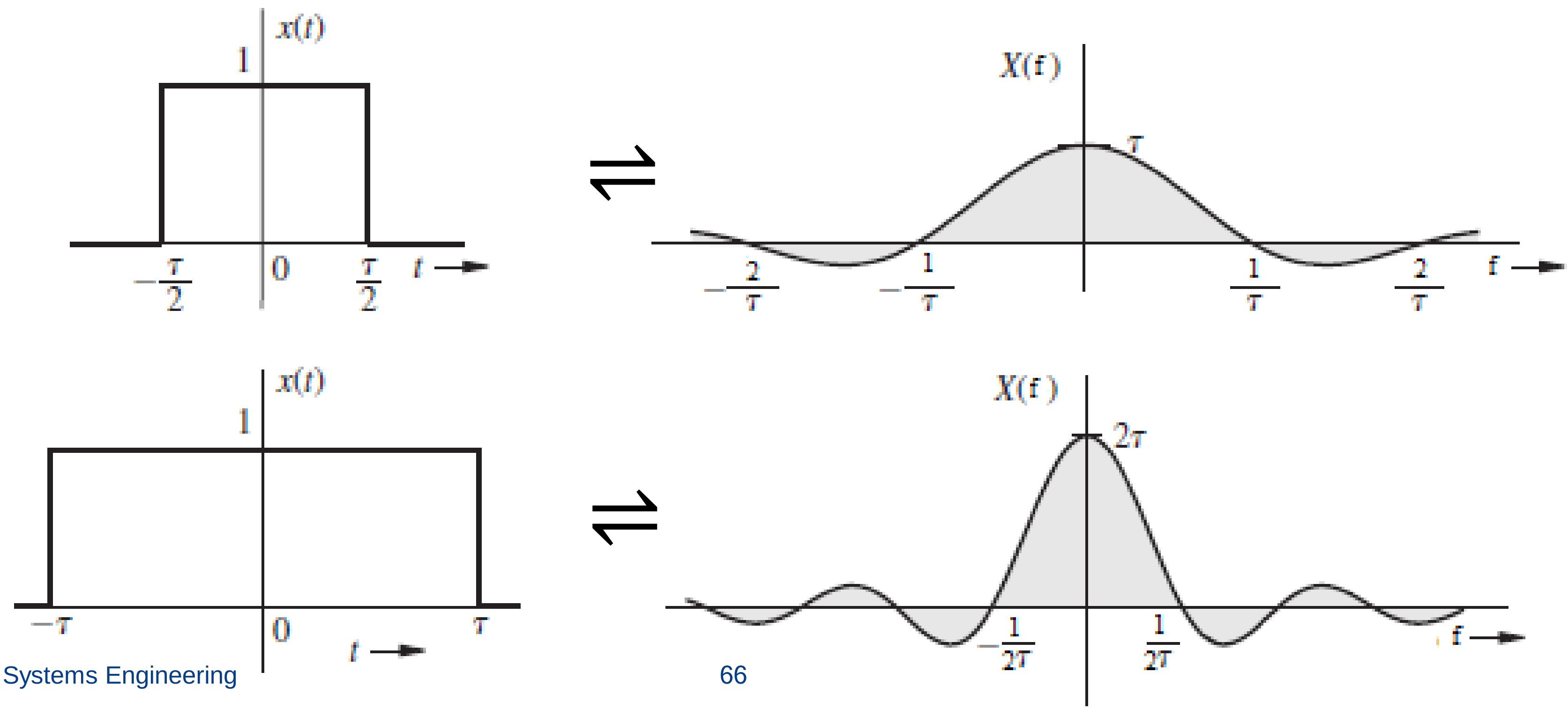
Reflection (inversion) property

$$x(-t) \rightleftharpoons X(-f)$$

SOME PROPERTIES OF THE FOURIER TRANSFORM

4. TIME SCALING:

The scaling property states that time compression of a signal results in its spectral expansion, and time expansion of the signal results in its spectral compression.



SOME PROPERTIES OF THE FOURIER TRANSFORM

4. TIME SCALING:

Signal Duration and Bandwidth

- The scaling property implies that if x(t) is wider, its spectrum is narrower, and vice versa. Doubling the signal duration halves its bandwidth, and vice versa. This suggests that the bandwidth of a signal is inversely proportional to the signal duration or width (in seconds).

- The bandwidth of a rectangular pulse of width τ seconds is $1/\tau$ Hz.

Example:

Find the Fourier transform of the signal $x(t) = \frac{1}{a+jt}$

Solution:

$$x(t) \rightleftharpoons X(f)$$

$$X(t) \rightleftharpoons x(-f)$$

$$x(t) \rightleftharpoons X(f)$$

$$x(at) \rightleftharpoons \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$$e^{-at}u(t) \rightleftharpoons \frac{1}{a+j2\pi f}$$

$$\frac{1}{a+j2\pi t} \rightleftharpoons e^{af}u(-f)$$

$$\frac{1}{a+j2\pi \frac{t}{2\pi}} \rightleftharpoons 2\pi e^{af}u(-2\pi f)$$

$$\frac{1}{a+jt} \rightleftharpoons 2\pi e^{af}u(-2\pi f)$$

Example:

Find the Fourier transform of the signal $x(t) = e^{at}u(-t)$

Solution:

$$x(t) \rightleftharpoons X(f)$$

$$x(-t) \rightleftharpoons X(-f)$$

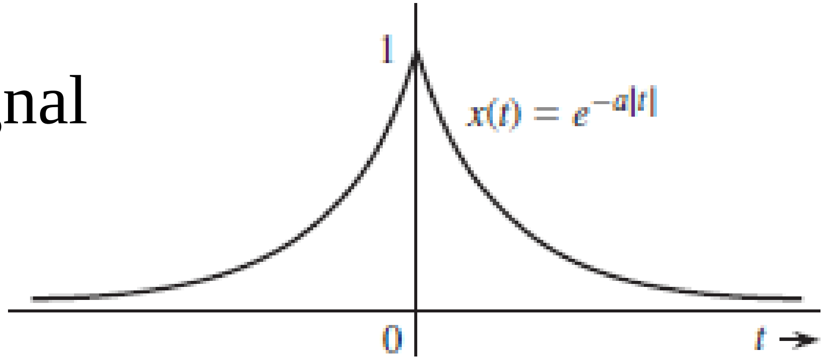
$$e^{-at}u(t) \rightleftharpoons \frac{1}{a + j2\pi f}$$

$$e^{at}u(-t) \rightleftharpoons \frac{1}{a - j2\pi f}$$

Example:

Find the Fourier transform of the signal

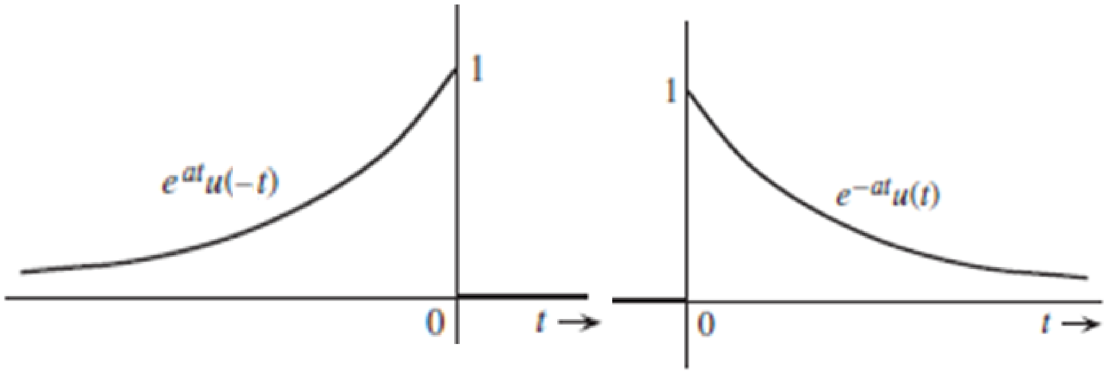
$$x(t) = e^{-a|t|}$$



Solution:

$$x(t) = e^{-a|t|}$$

$$x(t) = e^{at}u(-t) + e^{-at}u(t)$$



Solution:

$$x(t) \rightleftharpoons X(f)$$

$$x(-t) \rightleftharpoons X(-f)$$

$$e^{-at}u(t) \rightleftharpoons \frac{1}{a+j2\pi f}$$

$$+$$

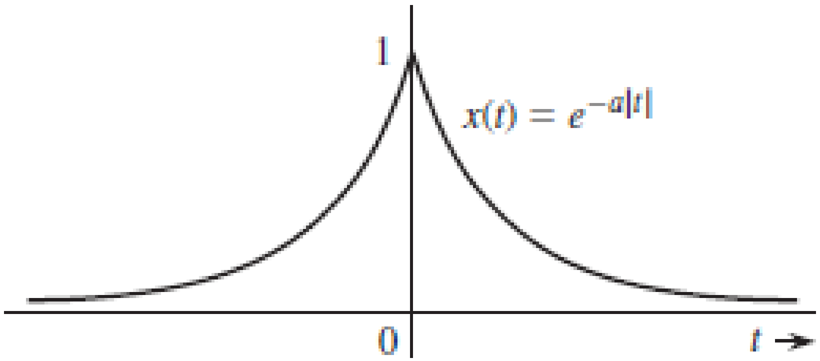
$$e^{at}u(-t) \rightleftharpoons \frac{1}{a-j2\pi f}$$

$$e^{-a|t|} \rightleftharpoons \frac{1}{a+j2\pi f}+\frac{1}{a-j2\pi f}$$

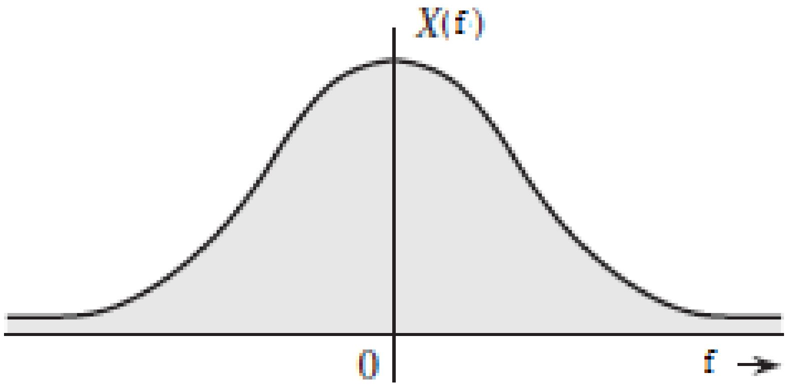
$$e^{-a|t|} \rightleftharpoons \frac{2a}{(a+j2\pi f)(a-j2\pi f)}$$

$$e^{-a|t|} \rightleftharpoons \frac{2a}{a^2+(2\pi f)^2}$$

Solution:



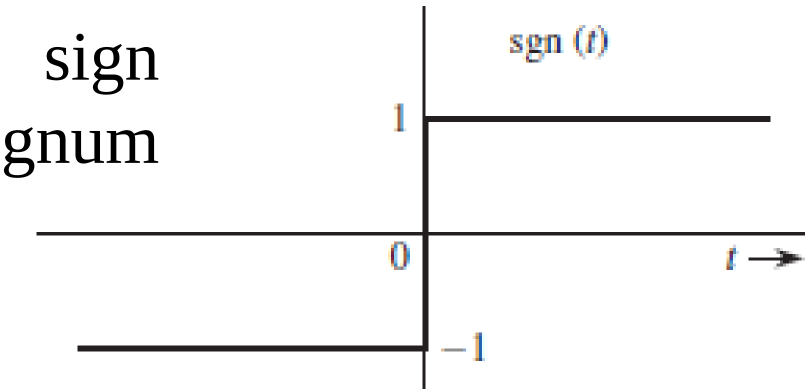
$\rightleftharpoons$



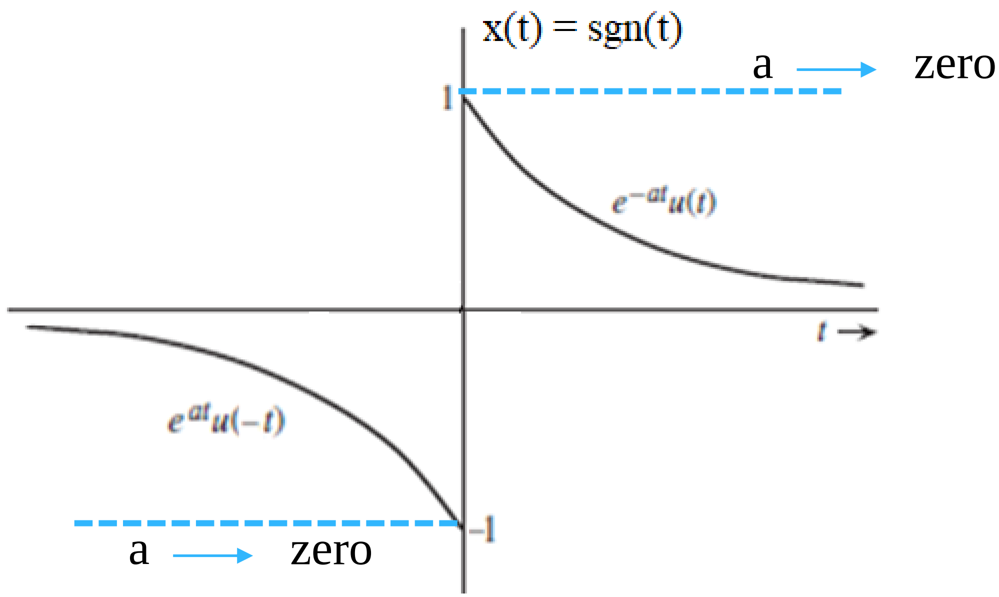
$$e^{-a|t|} \quad \rightleftharpoons \quad \frac{2a}{a^2 + (2\pi f)^2}$$

Example:

Find the Fourier transform of the sign function x(t) = sgn(t) [pronounced signum (t)], depicted in figure.



Solution:



$$sgn(t) = \lim_{a\to 0}\left(e^{at}u(-t) - e^{-at}u(t)\right)$$

Solution: $sgn(t) = \lim_{a \to 0}\left(e^{-at}u(t) - e^{at}u(-t)\right)$

$x(t) \rightleftharpoons X(f)$

$x(-t) \rightleftharpoons X(-f)$

$$e^{-at}u(t) \rightleftharpoons \frac{1}{a + j2\pi f}$$

$$-$$

$$e^{at}u(-t) \rightleftharpoons \frac{1}{a - j2\pi f}$$

$$e^{-at}u(t) - e^{at}u(-t) \rightleftharpoons \frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f}$$

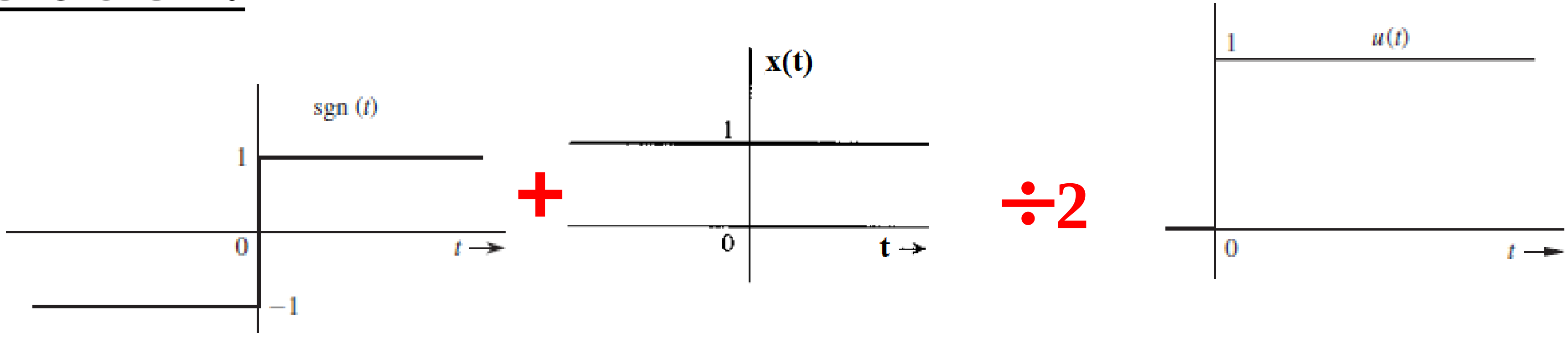
$$\lim_{a \to 0}\left(e^{-at}u(t) - e^{at}u(-t)\right) \rightleftharpoons \frac{1}{j2\pi f} + \frac{1}{j2\pi f}$$

$$sgn(t) \rightleftharpoons \frac{1}{j\pi f}$$

Example:

Find the Fourier transform of the unit step function x(t) = u(t).

Solution:



$$u(t) = \frac{1}{2} + \frac{1}{2} sgn(t)$$

$$u(t) \rightleftharpoons \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

SOME PROPERTIES OF THE FOURIER TRANSFORM

5. TIME SHIFT:

The time shift property states that if

$$x(t) \rightleftharpoons X(f)$$

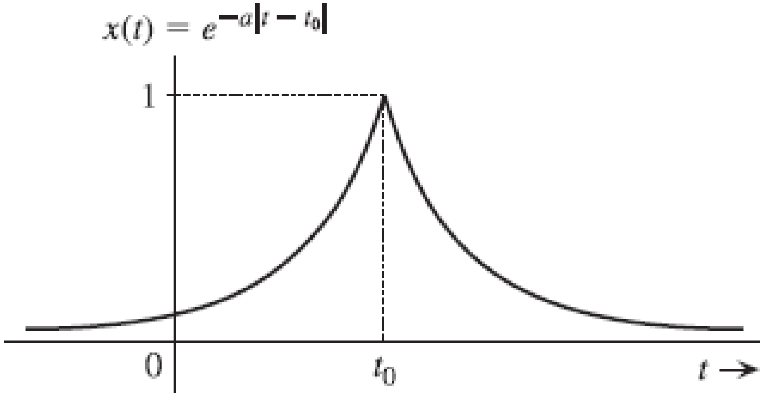
Then,

$$x(t - t_0) \rightleftharpoons X(f)e^{-j2\pi f t_0}$$

This result shows that delaying a signal by t_0 seconds does not change its amplitude spectrum. The phase spectrum, however, is changed by $-2\pi f t_0$.

Example:

Find the Fourier transform of the function $x(t) = e^{-a|t-t_0|}$.



Solution:

$$x(t) \rightleftharpoons X(f)$$

$$x(-t) \rightleftharpoons X(-f)$$

$$e^{-at}u(t) \rightleftharpoons \frac{1}{a + j2\pi f}$$

+

$$e^{at}u(-t) \rightleftharpoons \frac{1}{a - j2\pi f}$$

$$e^{-a|t|} \rightleftharpoons \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f}$$

$$e^{-a|t|} \rightleftharpoons \frac{2a}{a^2 + (2\pi f)^2}$$

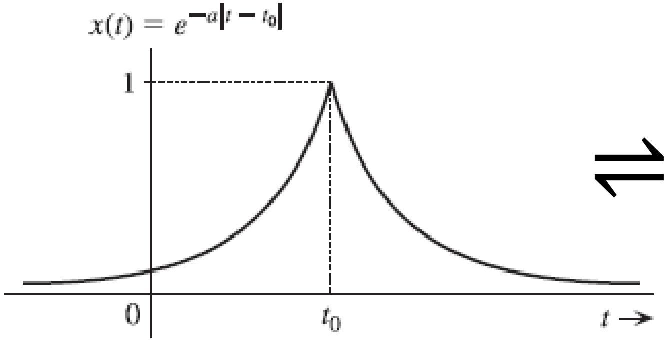
Solution:

$$e^{-a|t|} \rightleftharpoons \frac{2a}{a^2 + (2\pi f)^2}$$

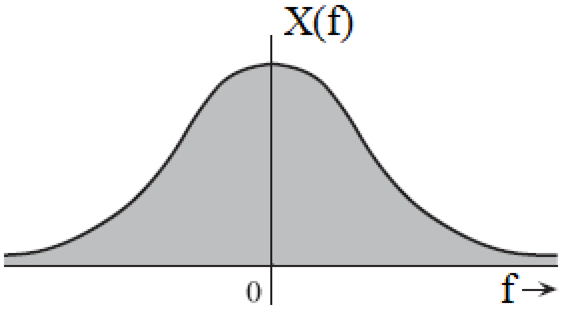
$$x(t) \rightleftharpoons X(f)$$

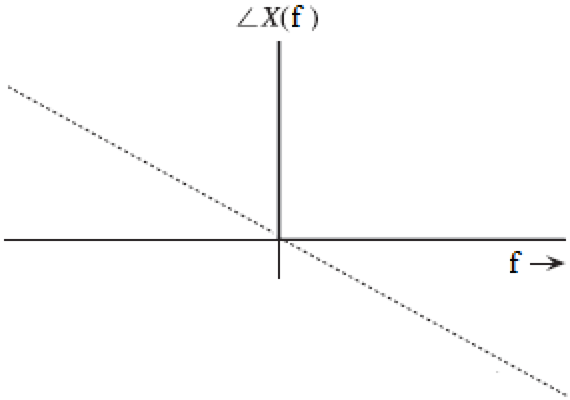
$$x(t - t_0) \rightleftharpoons X(f)e^{-j2\pi f t_0}$$

$$e^{-a|t-t_0|} \rightleftharpoons \frac{2a}{a^2 + (2\pi f)^2} e^{-j2\pi f t_0}$$



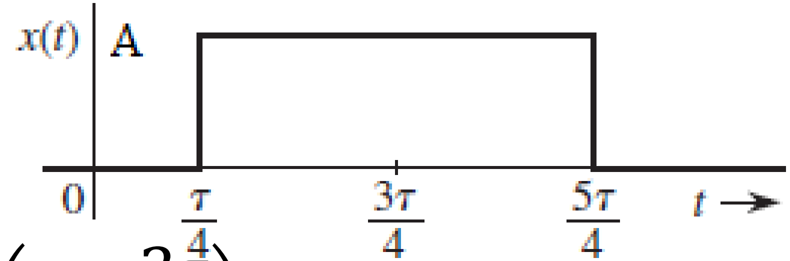
$\rightleftharpoons$





Example:

Find the Fourier transform of the function $x(t)$ shown in figure.



Solution:

$$x(t) = A\, rect\left(\frac{t - \frac{3\tau}{4}}{\tau}\right)$$

$$x(t) \rightleftharpoons X(f)$$

$$x(t - t_0) \rightleftharpoons X(f) e^{-j2\pi f t_0}$$

$$A\, rect\left(\frac{t}{\tau}\right) \rightleftharpoons A\tau\, Sa(\pi f \tau)$$

$$A\, rect\left(\frac{t - \frac{3\tau}{4}}{\tau}\right) \rightleftharpoons A\tau\, Sa(\pi f \tau) e^{-j2\pi f \frac{3\tau}{4}}$$

SOME PROPERTIES OF THE FOURIER TRANSFORM

6. FREQUENCY-SHIFT

The frequency shift property states that if

$$x(t) \rightleftharpoons X(f)$$

Then,

$$x(t)e^{j2\pi f_0 t} \rightleftharpoons X(f - f_0)$$

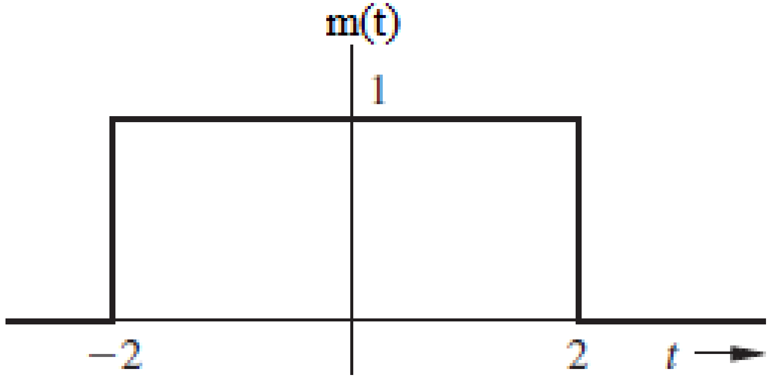
According to this property, the multiplication of a signal by a factor $e^{j2\pi f_0 t}$ shifts the spectrum of that signal by f_0.

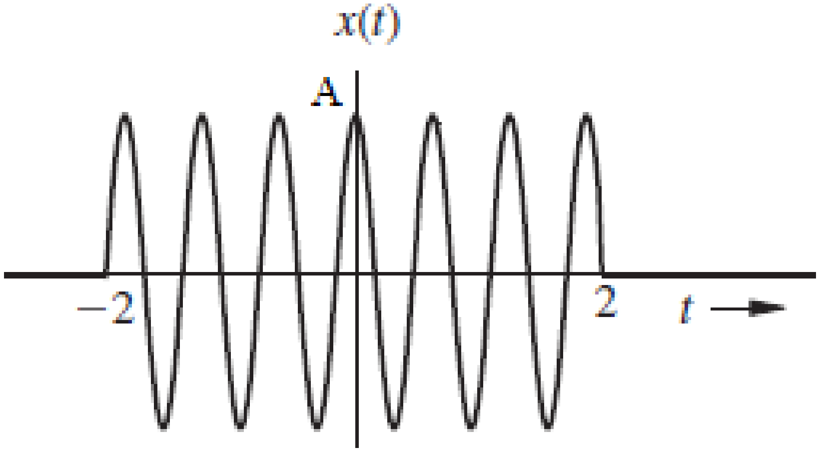
Example:

Find and sketch the Fourier transform of the **amplitude modulated** AM signal,

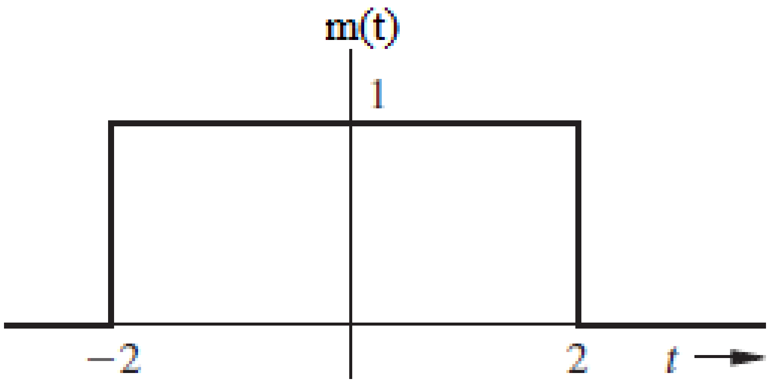
$$x(t) = A\,m(t)\cos(2\pi f_0 t)$$

where the **modulating signal** is $m(t) = rect\left(\frac{t}{4}\right)$, and the carrier is $\mathrm{c(t)} = A\ \cos(2\pi f_0 t)$

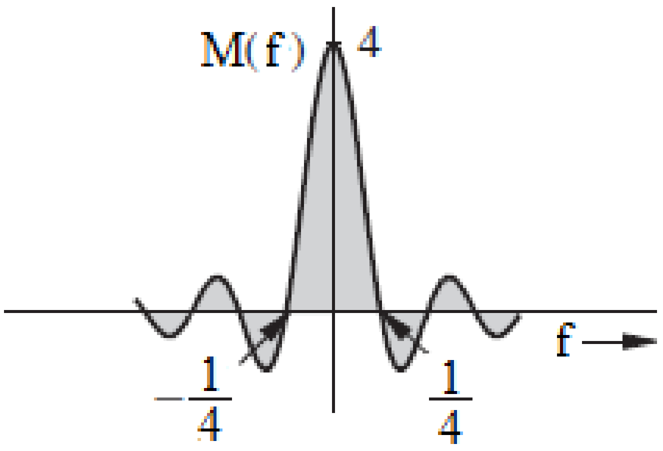




Solution:



$$\rightleftharpoons$$



$$m(t) = rect\left(\frac{t}{4}\right)$$

$$A\, rect\left(\frac{t}{\tau}\right) \rightleftharpoons A\tau\, Sa(\pi f \tau)$$

$$m(t) = rect\left(\frac{t}{4}\right) \rightleftharpoons M(f) = 4\, Sa(4\pi f)$$

Solution:

$$x(t) = A\, m(t) \cos(2\pi f_0 t)$$

$$x(t) = \frac{A}{2} m(t)\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right)$$

$$x(t) = \frac{A}{2} m(t) e^{j2\pi f_0 t} + \frac{A}{2} m(t) e^{-j2\pi f_0 t}$$

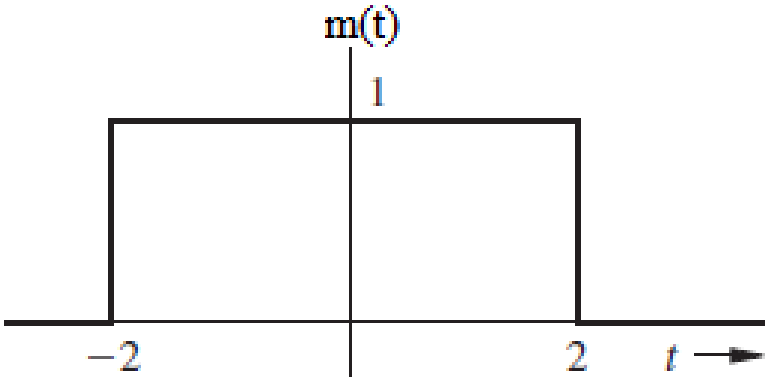
$$X(f) = \frac{A}{2} M(f - f_0) + \frac{A}{2} M(f + f_0)$$

$$M(f) = 4\, Sa(4\pi f)$$

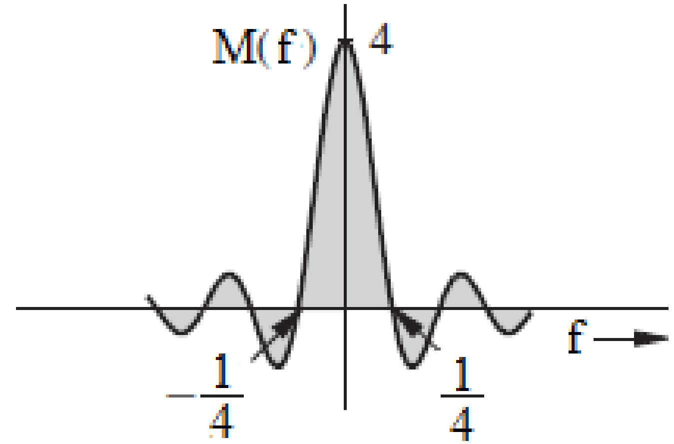
$$X(f) = 2ASa\big(4\pi(f - f_0)\big) + 2ASa\big(4\pi(f + f_0)\big)$$

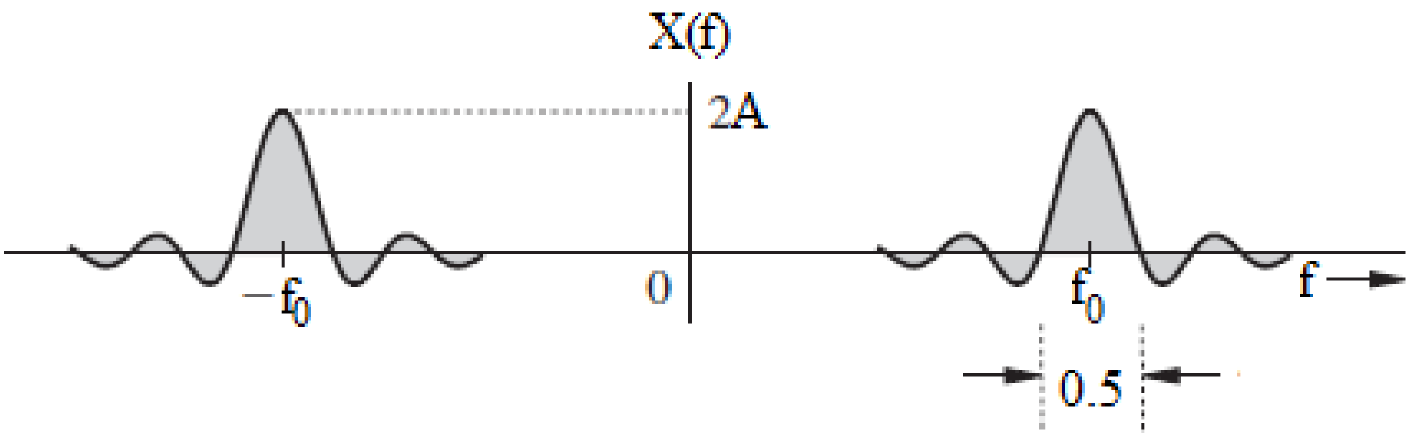
Solution:

$$X(f) = 2ASa\big(4\pi(f - f_0)\big) + 2ASa\big(4\pi(f + f_0)\big)$$



$\rightleftharpoons$

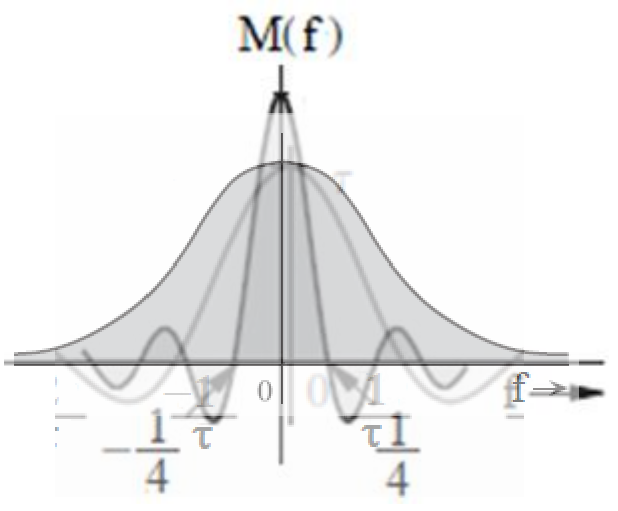




APPLICATIONS OF MODULATION

Modulation is used to shift signal spectra.

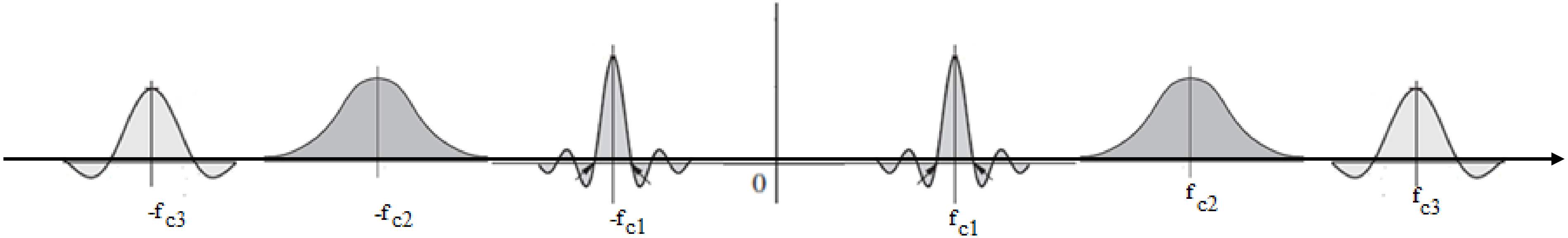
If several signals, all occupying the same frequency band, are transmitted simultaneously over the same transmission medium, they will all interfere; it will be impossible to separate or retrieve them at a receiver. For example, if all radio stations decide to broadcast audio signals simultaneously, a receiver will not be able to separate them.



APPLICATIONS OF MODULATION

Modulation is used to shift signal spectra.

The problem is solved by using modulation, whereby each radio station is assigned a distinct carrier frequency. Each station transmits a modulated signal. This procedure shifts the signal spectrum to its allocated band, which is not occupied by any other station. A radio receiver can pick up any station by tuning to the band of the desired station. This is called frequency division multiplexing **(FDM)**.



SOME PROPERTIES OF THE FOURIER TRANSFORM

7. Convolution:

The time-convolution property, state that if

$$x_1(t) \rightleftharpoons X_1(f) \qquad x_2(t) \rightleftharpoons X_2(f)$$

Then,

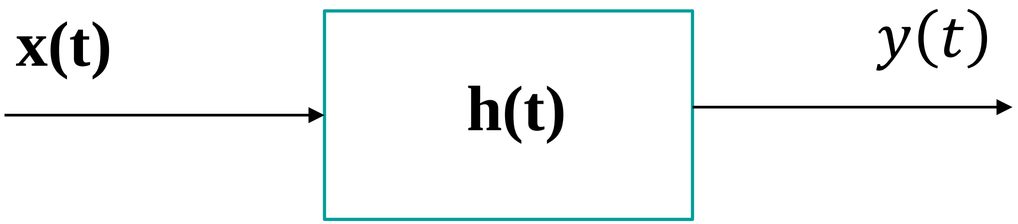
$$x_1(t) * x_2(t) \rightleftharpoons X_1(f)X_2(f)$$

The frequency-convolution property, state that if

$$x_1(t)x_2(t) \rightleftharpoons X_1(f) * X_2(f)$$

Example:

For an LTIC system with the unit impulse response $h(t) = e^{-2t}u(t)$ determine the response y(t) for the input $x(t) = e^{-t}u(t)$



Solution:

$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f)H(f)$$

$$e^{-at}u(t) \rightleftharpoons \frac{1}{a + j2\pi f}$$

$$x(t) = e^{-t}u(t) \rightleftharpoons X(f) = \frac{1}{1 + j2\pi f}$$

Solution:

$$h(t) = e^{-2t}u(t) \rightleftharpoons H(f) = \frac{1}{2 + j2\pi f}$$

$$Y(f) = X(f)H(f)$$

$$Y(f) = \left(\frac{1}{1 + j2\pi f}\right)\left(\frac{1}{2 + j2\pi f}\right)$$

$$Y(f) = \frac{1}{(1 + j2\pi f)(2 + j2\pi f)}$$

$$Y(f) = \frac{K_1}{1 + j2\pi f} + \frac{K_2}{2 + j2\pi f}$$

where K_1, and K_2 are constants.

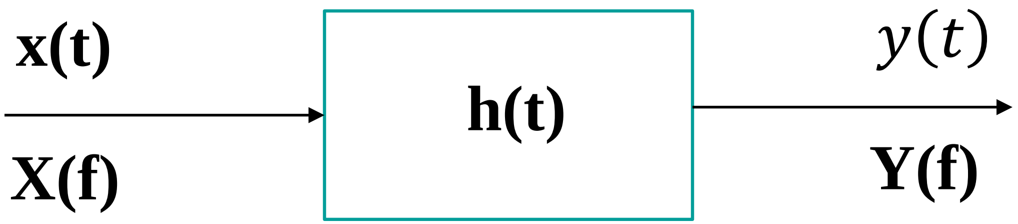
$$Y(f) = \frac{1}{1 + j2\pi f} - \frac{1}{2 + j2\pi f}$$

Inverse Fourier Transform

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

Example:

For an LTIC system with the transfer function $H(f) = \frac{1}{1+j2\pi f}$ determine the response y(t) for the input $x(t) = u(t)$.



Solution:

$$Y(f) = X(f)H(f)$$

$$u(t) \rightleftharpoons \frac{1}{2}\delta(f) + \frac{1}{j2\pi f}$$

$$Y(f) = \left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right)\frac{1}{1 + j2\pi f}$$

Solution:

$$Y(f) = \left(\frac{1}{2}\delta(f) + \frac{1}{j2\pi f}\right)\frac{1}{1 + j2\pi f}$$

$$Y(f) = \frac{1}{2}\delta(f) \times \frac{1}{1 + j2\pi f} + \frac{1}{j2\pi f} \times \frac{1}{1 + j2\pi f}$$

$$Y(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f} \times \frac{1 + j2\pi f - j2\pi f}{1 + j2\pi f}$$

$$Y(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f} \times \left(1 - \frac{j2\pi f}{1 + j2\pi f}\right)$$

$$Y(f) = \frac{1}{2}\delta(f) + \frac{1}{j2\pi f} - \frac{1}{1 + j2\pi f}$$

Take the inverse FT

$$y(t) = u(t) - e^{-t}u(t)$$

$$y(t) = (1 - e^{-t})u(t)$$

SOME PROPERTIES OF THE FOURIER TRANSFORM

8. TIME DIFFERENTIATION AND TIME INTEGRATION

The time-differentiation property, state that if

$$x(t) \rightleftharpoons X(f)$$

Then,

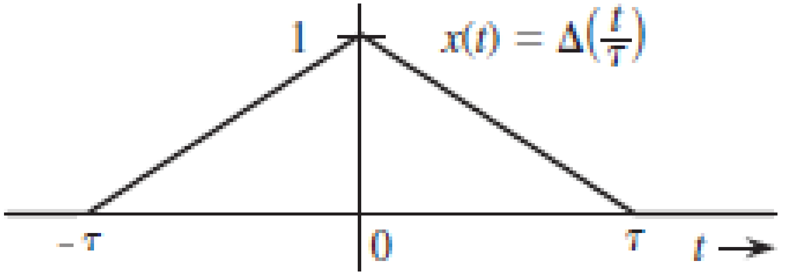
$$\frac{dx(t)}{dt} \rightleftharpoons j2\pi f X(f)$$

The time integration property, state that if

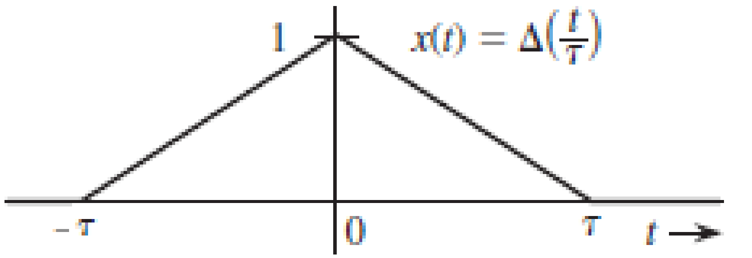
$$\int_{-\infty}^{t} x(\tau)d\tau \rightleftharpoons \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$$

Example:

Find the Fourier transform of the triangle pulse $x(t) = tri\left(\frac{x}{\tau}\right)$ illustrated in figure.

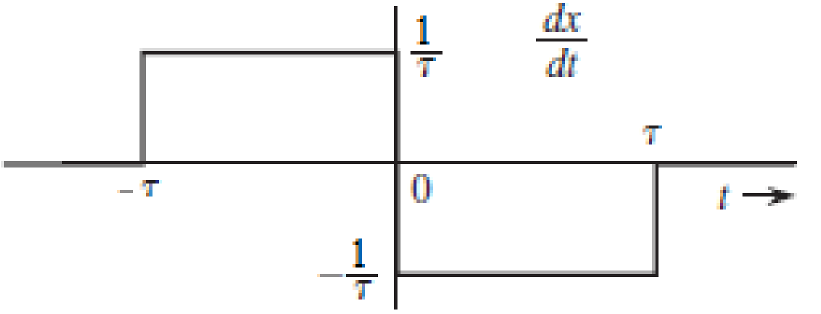


Solution:

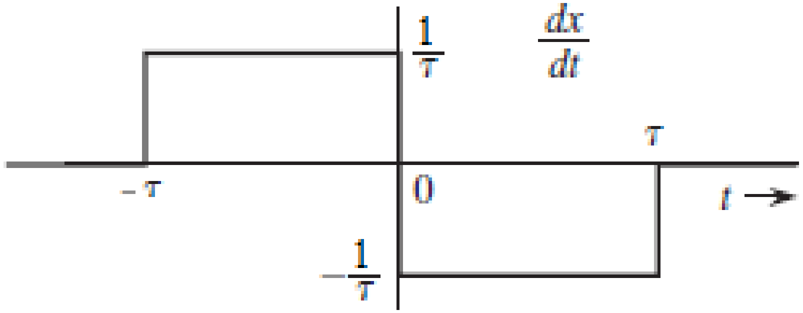


$$x(t) \rightleftharpoons X(f)$$

$$\frac{dx(t)}{dt} \rightleftharpoons j2\pi f X(f)$$



Solution:



$$\frac{dx(t)}{dt} = \frac{1}{\tau} rect\left(\frac{t+\frac{\tau}{2}}{\tau}\right) - \frac{1}{\tau} rect\left(\frac{t-\frac{\tau}{2}}{\tau}\right)$$

$$\frac{dx(t)}{dt} \rightleftharpoons j2\pi f X(f)$$

$$j2\pi f X(f) = \frac{1}{\tau} \times \tau\, Sa(\pi f \tau) e^{j2\pi f\frac{\tau}{2}} - \frac{1}{\tau} \times \tau\, Sa(\pi f \tau) e^{-j2\pi f\frac{\tau}{2}}$$

$$j2\pi f X(f) = Sa(\pi f \tau)\left(e^{j2\pi f\frac{\tau}{2}} - e^{-j2\pi f\frac{\tau}{2}}\right)$$

Solution:

$$j2\pi f X(f) = Sa(\pi f \tau)\left(e^{j2\pi f\frac{\tau}{2}} - \ e^{-j2\pi f\frac{\tau}{2}}\right)$$

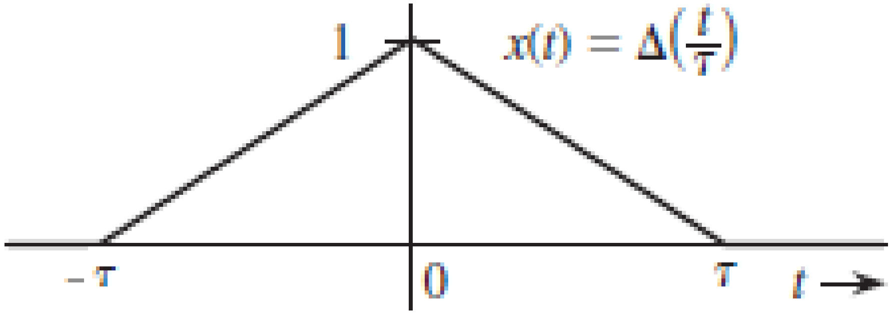
$$X(f) = \frac{Sa(\pi f \tau)\left(e^{j\pi f \tau} - \ e^{-j\pi f \tau}\right)}{j2\pi f}$$

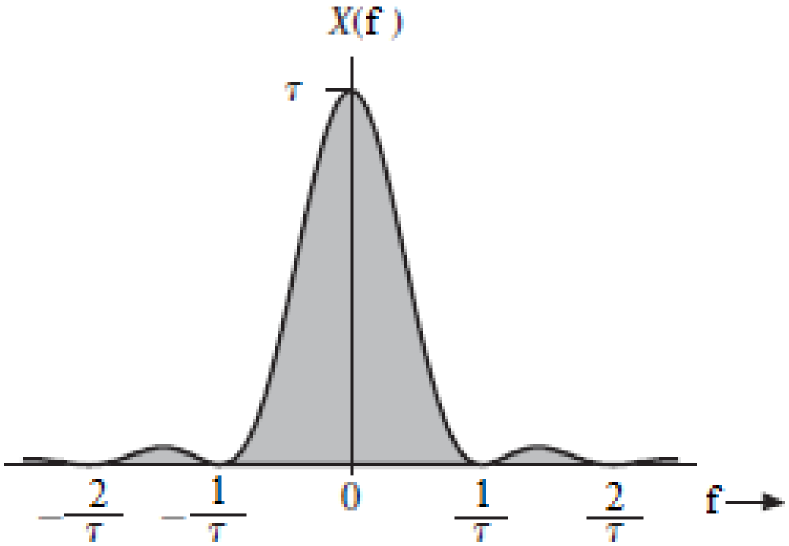
$$X(f) = \frac{Sa(\pi f \tau)\sin(\pi f \tau)}{\pi f}$$

$$X(f) = \tau \frac{Sa(\pi f \tau)\sin(\pi f \tau)}{\pi f \tau}$$

$$X(f) = \tau\, Sa^2(\pi f \tau)$$

Solution:





$$A\ tri\left(\frac{x}{\tau}\right) \rightleftharpoons A\tau\ Sa^2(\pi f\tau)$$

Fourier Transform Properties

Operation	$\boldsymbol{x(t)}$	$\boldsymbol{X(\omega)}$
Scalar multiplication	$kx(t)$	$kX(\mathrm{f})$
Addition	$x_1(t) + x_2(t)$	$X_1(\mathrm{f}) + X_2(\mathrm{f})$
Conjugation	$x^*(t)$	$X^*(-\mathrm{f})$
Duality	$X(t)$	$x(-\mathrm{f})$
Scaling (a real)	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{\mathrm{f}}{a}\right)$
Time shifting	$x(t - t_0)$	$X(\mathrm{f})e^{-j2\pi \mathrm{f} \mathrm{t}_0}$
Frequency shifting	$x(t)\, e^{j2\pi \mathrm{f}_0 \mathrm{t}}$	$X(\mathrm{f} - \mathrm{f}_0)$
Time convolution	$x_1(t) * x_2(t)$	$X_1(\mathrm{f})X_2(\mathrm{f})$
Frequency convolution	$x_1(t)x_2(t)$	$X_1(\mathrm{f}) * X_2(\mathrm{f})$
Time differentiation	$\frac{d\,x(t)}{dt}$	$j2\pi \mathrm{f}\, X(\mathrm{f})$
Time integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(\mathrm{f})}{j2\pi \mathrm{f}} + \frac{1}{2} X(0)\delta(\mathrm{f})$

Short Table of Fourier Transform

$x(t)$	$X(f)$
1	$\delta(f)$
$\delta(1)$	1
$e^{-at}u(t)$	$\frac{1}{a+j2\pi f}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$
$\cos 2\pi f_0 t$	$\frac{1}{2}\left(\delta(f-f_0)+\delta(f+f_0)\right)$
$\sin 2\pi f_0 t$	$\frac{1}{2j}\left(\delta(f-f_0)-\delta(f+f_0)\right)$

Short Table of Fourier Transform

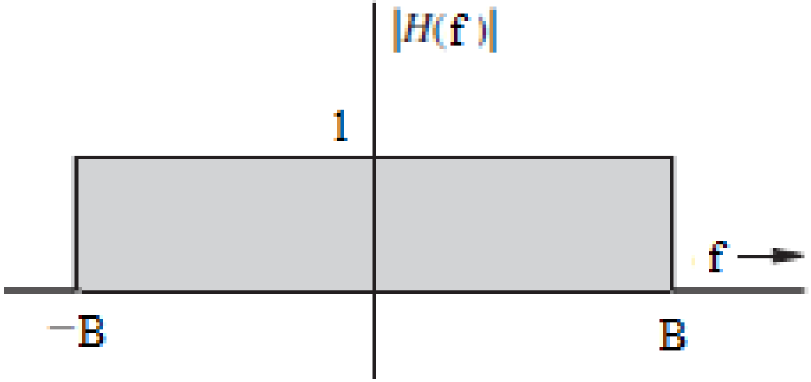
$x(t)$	$X(f)$
$A\, rect\left(\frac{t}{\tau}\right)$	$A\tau Sa(\pi f \tau)$
$A\, tri\left(\frac{t}{\tau}\right)$	$A\tau Sa^2(\pi f \tau)$

Ideal Filters

Ideal filters allow distortionless transmission of a certain band of frequencies and completely suppress the remaining frequencies.

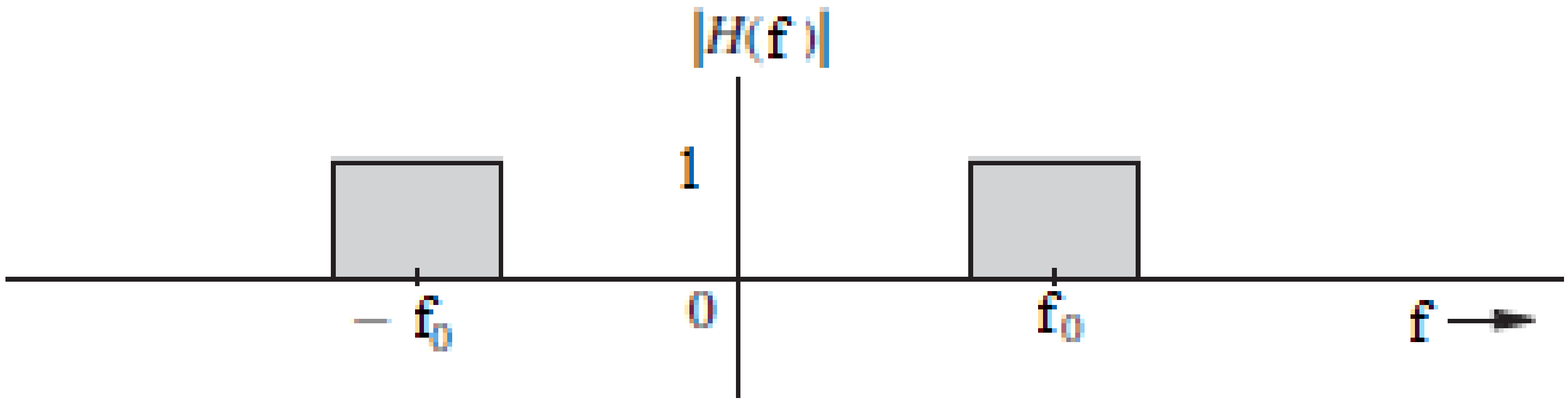
Low pass Filter LPF

The ideal low pass filter allows all frequency components below f = B Hz to pass without distortion and suppresses all components above f =B.



Band pass Filter BPF

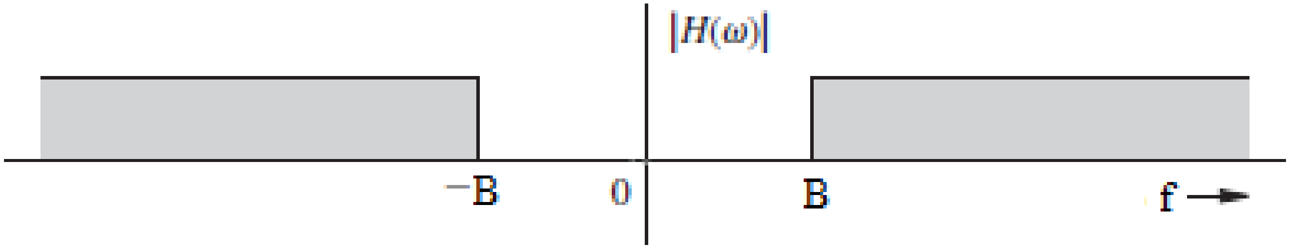
The ideal bandpass filter allows all frequency components in a band of frequency around its center frequency (f_0) to pass without distortion and suppresses all components outside that band.



Ideal Filters

High pass Filter HPF

The ideal high pass filter allows all frequency components above f = B Hz to pass without distortion and suppresses all components below f =B.

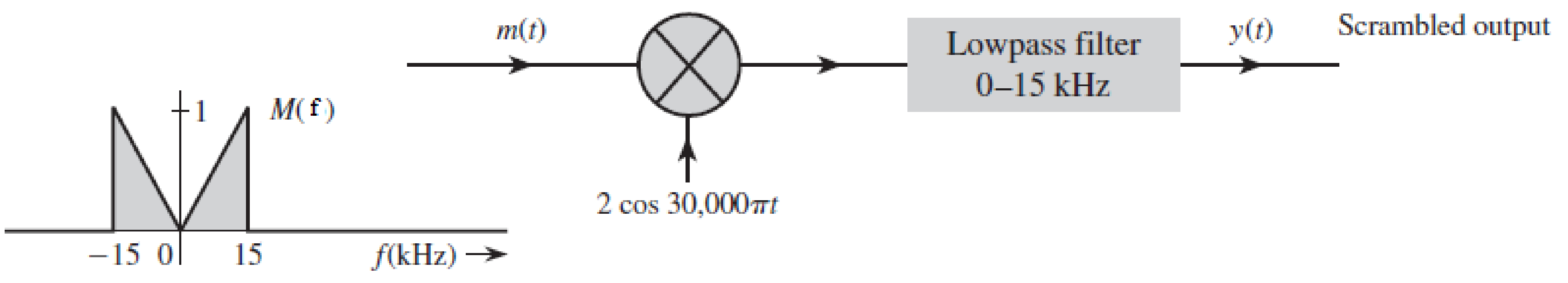


Example:

The system shown in figure is used for scrambling audio signals. The output y(t) is the scrambled version of the input m(t).

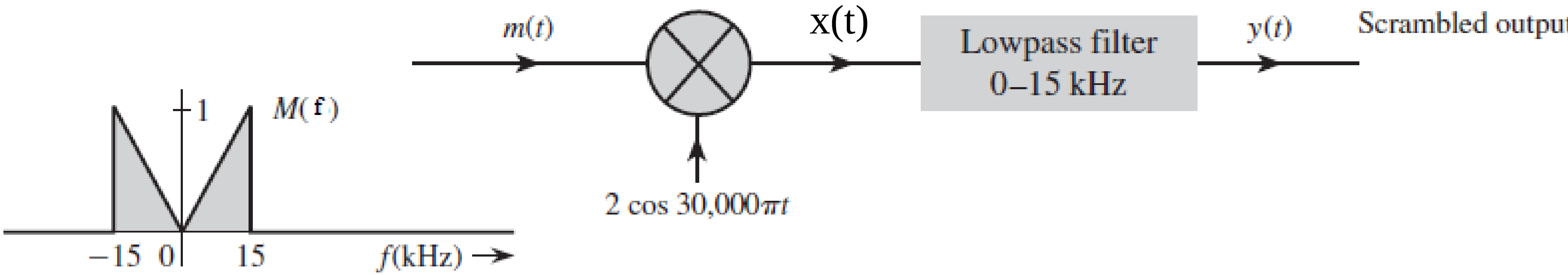
(a) Find the spectrum of the scrambled signal y(t).

(b) Suggest a method of descrambling y(t) to obtain m(t).



Solution:

a.



$$x(t) = 2\, m(t) \cos(30{,}000\pi t)$$

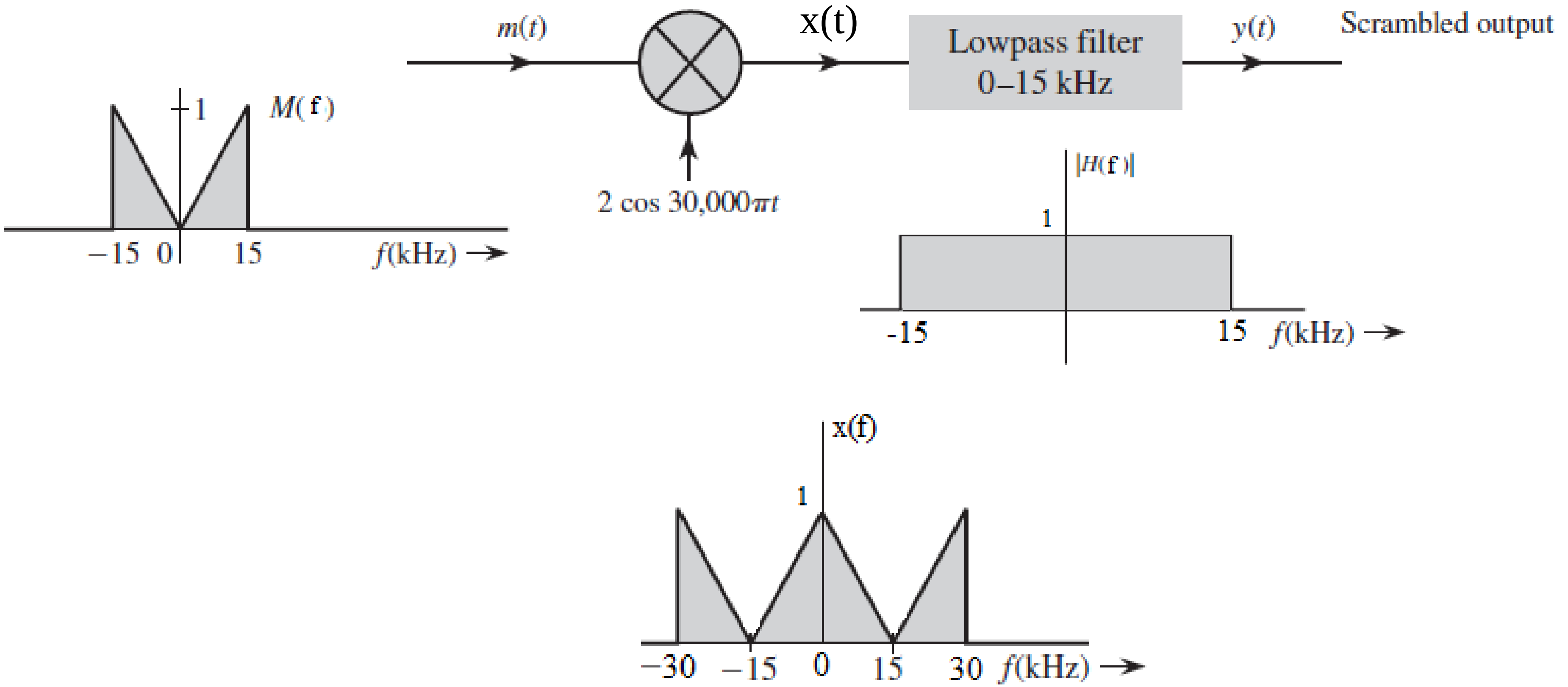
$$x(t) = m(t)\left(e^{j30{,}000\pi t} + e^{-j30{,}000\pi t}\right)$$

$$x(t) = m(t)e^{j30{,}000\pi t} + m(t)e^{-j30{,}000\pi t}$$

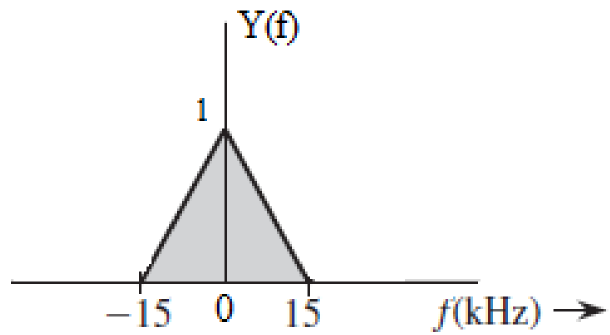
$$X(f) = M(f - 15{,}000) + M(f + 15{,}000)$$

Solution:

a.



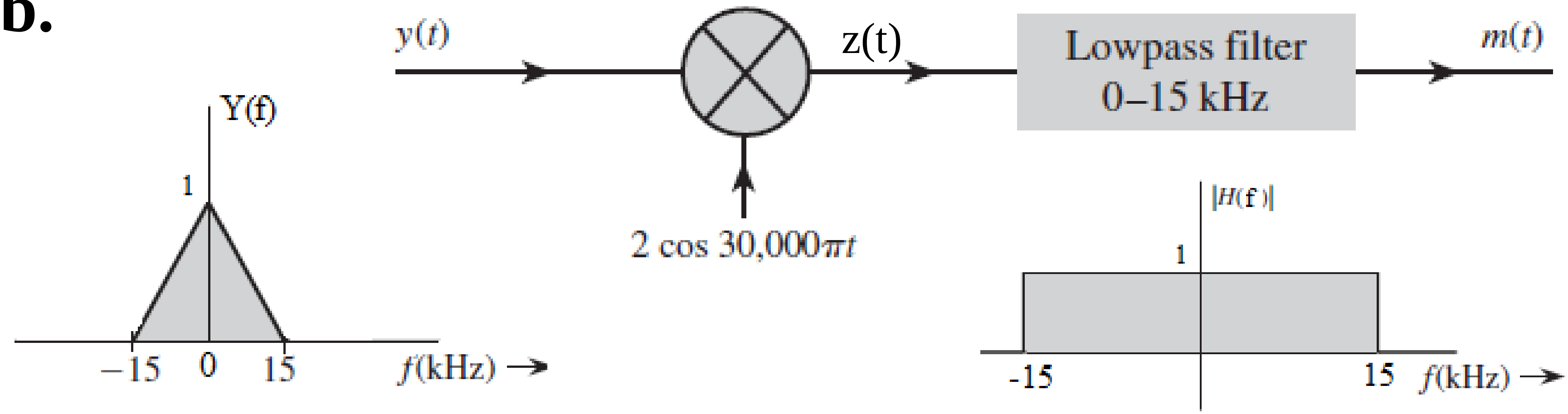
m(t)
x(t)
Lowpass filter
0–15 kHz
y(t)
Scrambled output
2 cos 30,000πt
M(f)
1
–15 0
15
f(kHz)
|H(f)|
1
-15
15
f(kHz)
x(f)
1
–30 –15 0 15 30 f(kHz)

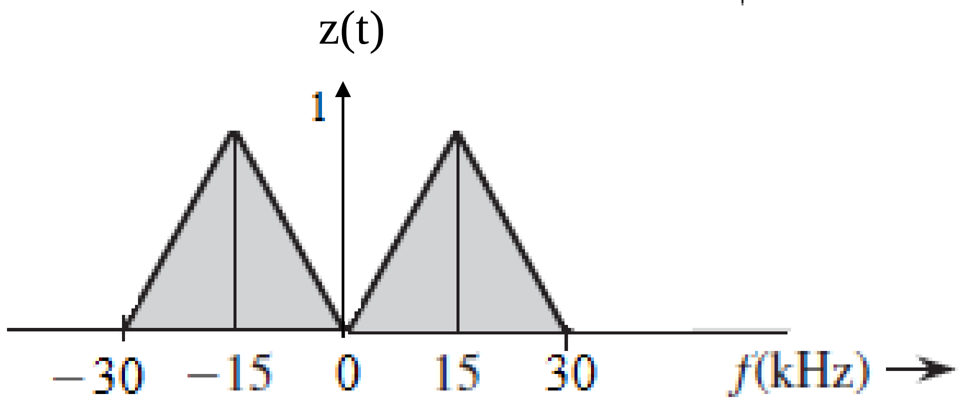


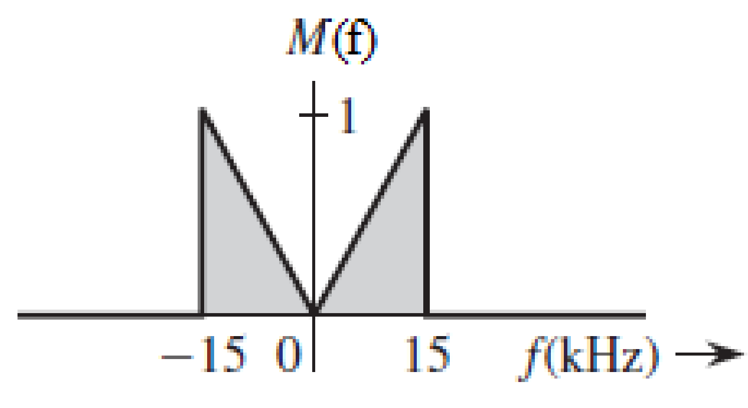
Y(f)
1
–15 0 15
f(kHz)

Solution:

b.


y(t)
z(t)
Lowpass filter
0–15 kHz
m(t)
2 cos 30,000πt
Y(f)
1
−15
0
15
f(kHz) →
|H(f)|
1
-15
15
f(kHz) →


z(t)
1
−30
−15
0
15
30
f(kHz) →


M(f)
1
−15
0
15
f(kHz) →

Signal Energy

Parseval's Theorem

Signal energy (E_x) of a signal x(t) is defined as the area under $|x(t)|^2$,

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

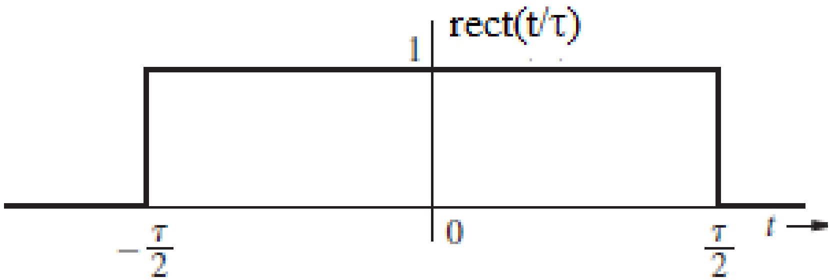
Signal energy can be related to the signal spectrum X(f) by:

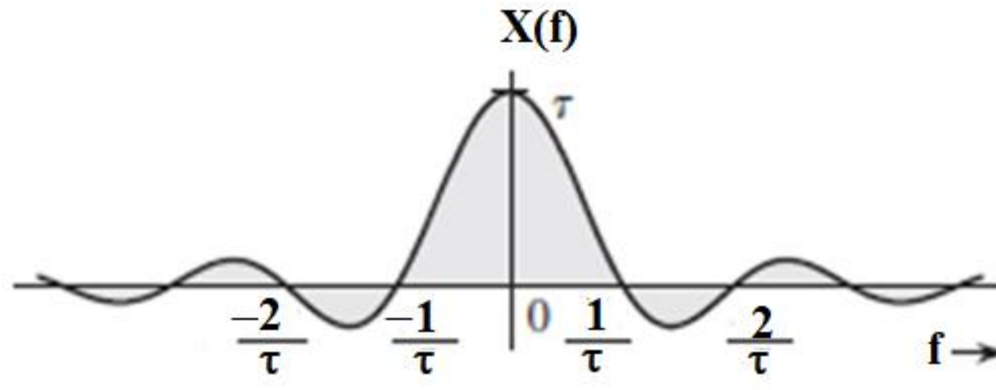
$$E_x = \int_{-\infty}^{\infty} |X(f)|^2 df$$

Example:

Using Parseval's theorem show that $\int_{-\infty}^{\infty} |Sa(x)|^2 dx = \pi$

Solution:





$$rect\left(\frac{t}{\tau}\right) \quad \rightleftharpoons \quad \tau\, Sa(\pi f \tau)$$

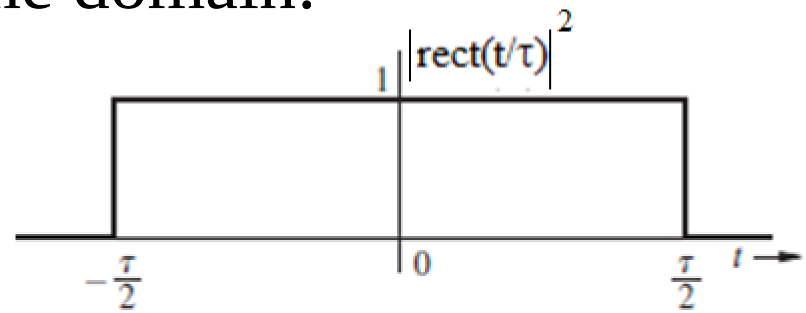
Let τ = 1/π, then;

$$rect(\pi t) \quad \rightleftharpoons \quad \frac{1}{\pi} Sa(f)$$

Solution:

Calculating the signal energy in time domain:

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \tau = \frac{1}{\pi}$$



Calculating the signal energy in frequency domain:

$$E_x = \int_{-\infty}^{\infty} |X(f)|^2 df = \frac{1}{\pi^2}\int_{-\infty}^{\infty} |Sa(f)|^2 df$$

Using Parseval's theorem:

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty} |Sa(f)|^2 df = \frac{1}{\pi}$$

Solution:

$$\int_{-\infty}^{\infty} |Sa(f)|^2 df = \pi$$

We can replace f with any dummy variable:

$$\int_{-\infty}^{\infty} |Sa(x)|^2 dx = \pi$$