Chapter 2

Time Domain analysis of continuous time systems

SYSTEM RESPONSE TO EXTERNAL INPUT

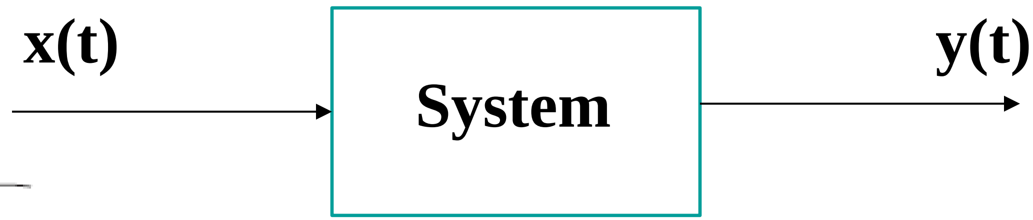
We will consider two methods of analysis of linear time-invariant (LTI) systems: the time-domain method and the frequency-domain method.

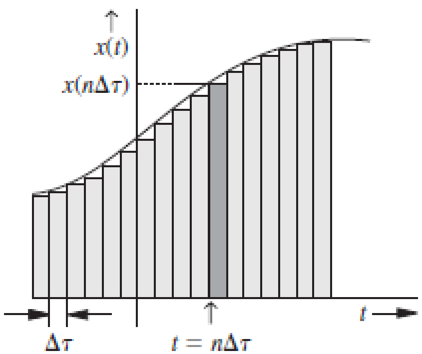
In this chapter we discuss the time-domain analysis of linear, time-invariant, continuous-time (LTIC) systems.

We will find the system response y(t) to an input x(t) when the system is in the zero state, that is, when all initial conditions are zero.

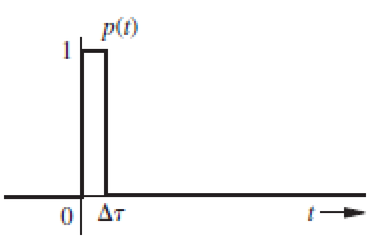
SYSTEM RESPONSE TO EXTERNAL INPUT

We shall use the superposition property for finding the system response to an arbitrary input x(t).





The input x(t) could be written as a sum of narrow rectangular pulses. The pulse starting at **t=nΔτ** has a height **x(nΔτ)** and a width **Δτ** as shown in figure.

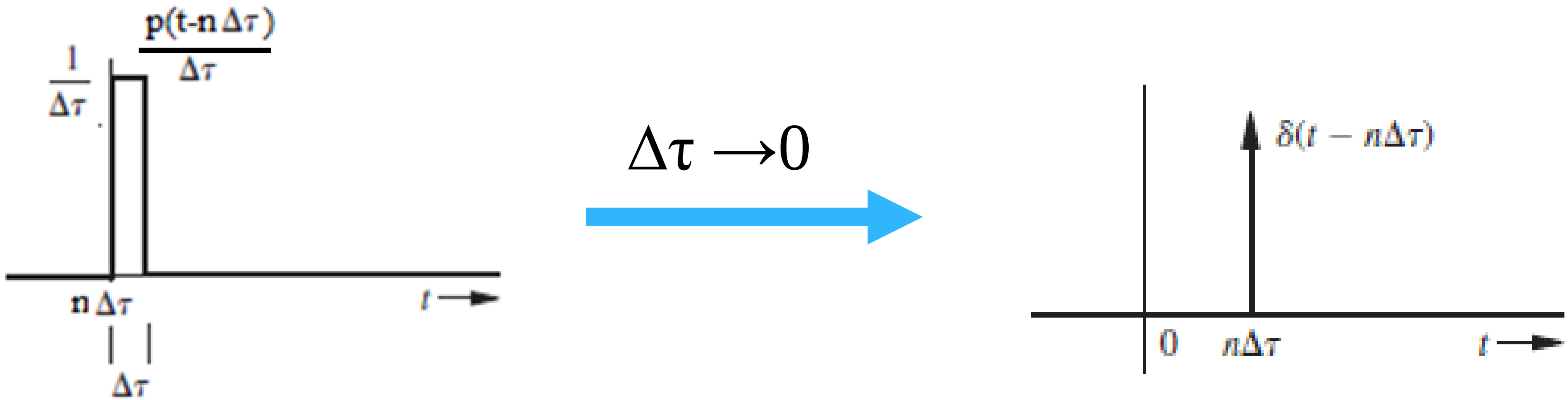


$$x(t) = \lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau) p(t - n\Delta\tau)$$

$$x(t) = \lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau) \left[\frac{p(t - n\Delta\tau)}{\Delta\tau} \right] \Delta\tau$$

SYSTEM RESPONSE TO EXTERNAL INPUT

As **Δτ →0**, the height of this strip→∞, but its area remains 1. Hence, this strip approaches an impulse δ(t−nτ) as τ →0.

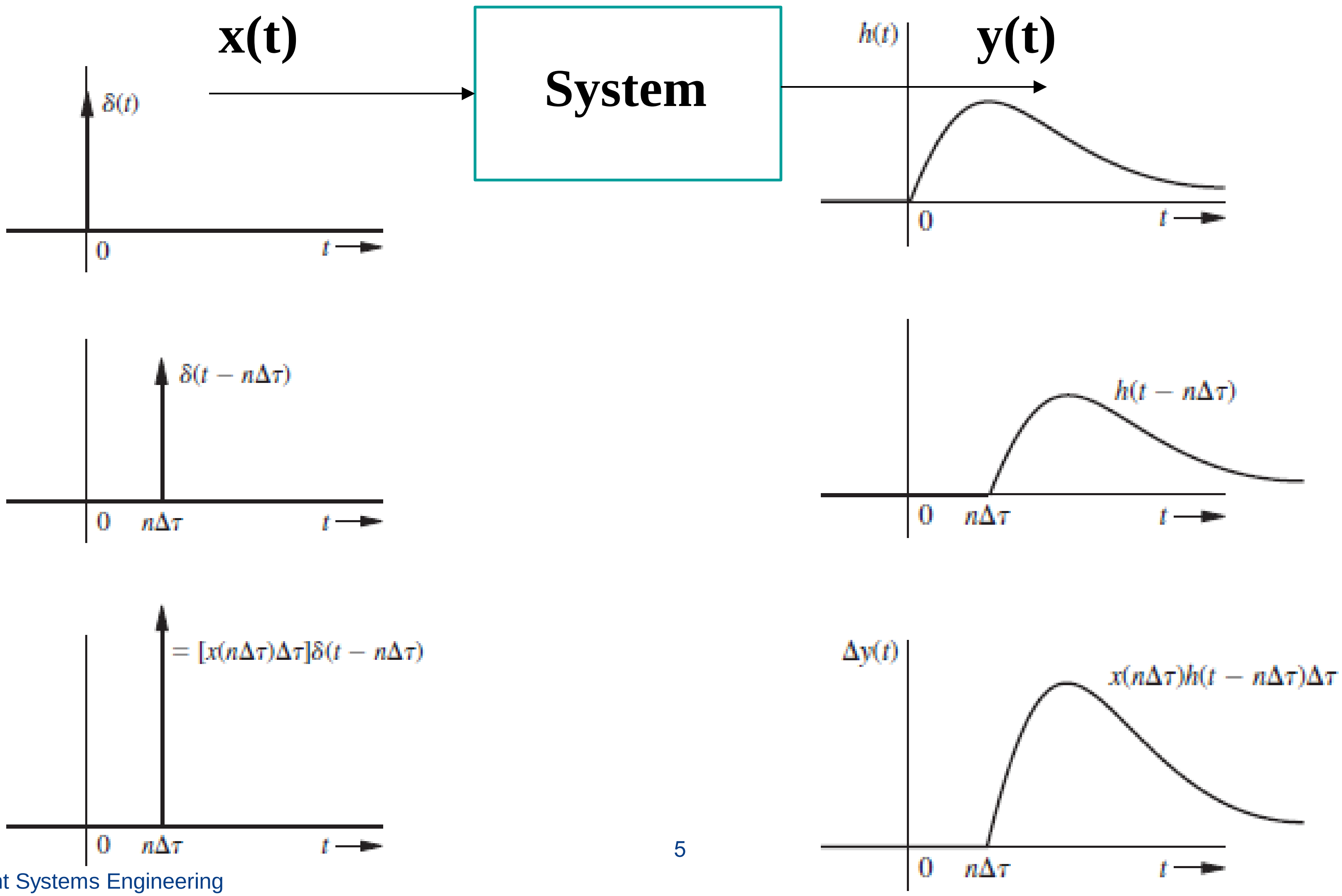


The input x(t) could be written as:

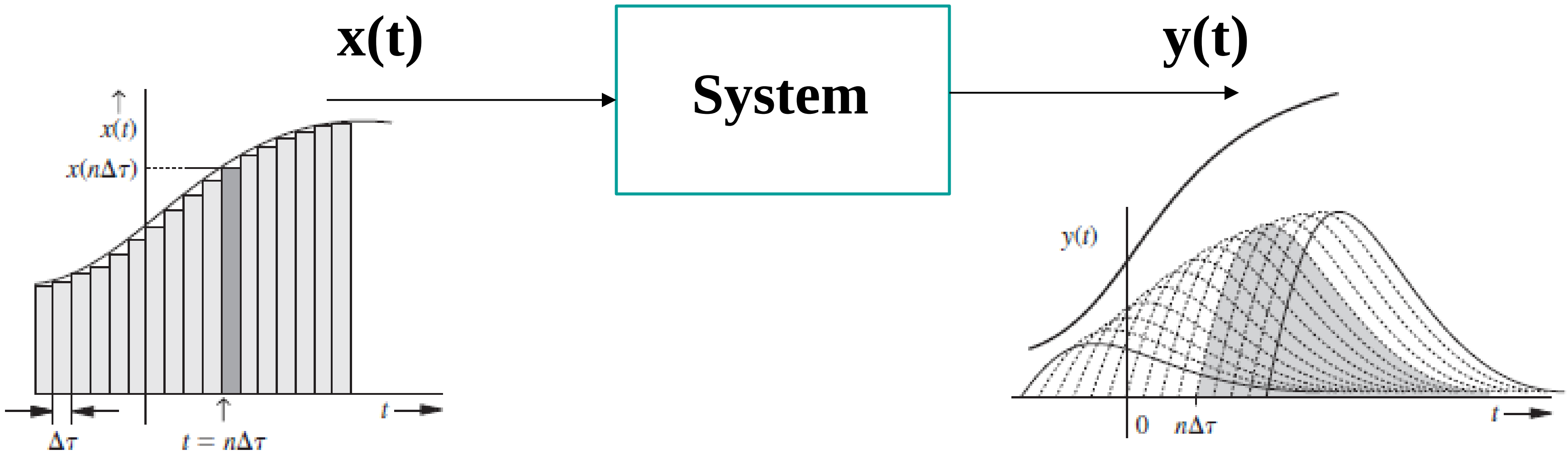
$$x(t) = \lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)\delta(t - n\Delta\tau)\Delta\tau$$

SYSTEM RESPONSE TO EXTERNAL INPUT

To find the response for this input **x(t)**, we consider the input and the corresponding output pairs:



SYSTEM RESPONSE TO EXTERNAL INPUT



So, the system response y(t) could be expressed as:

$$y(t) = \lim_{\Delta\tau \to 0} \sum_{\tau} x(n\Delta\tau)h(t - n\Delta\tau)\Delta\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

where h(t) is the impulse response of the system.

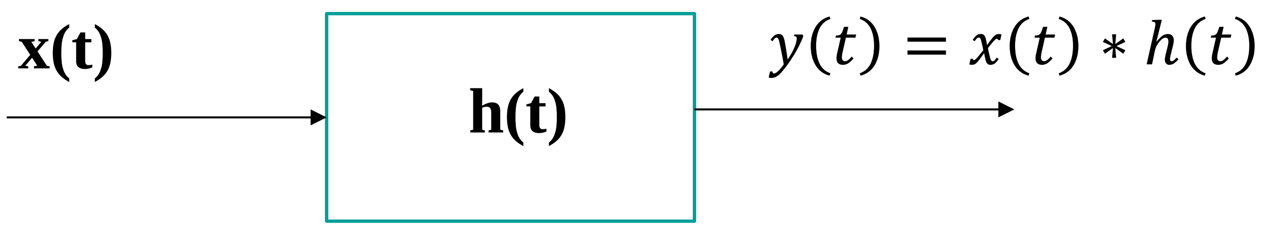
The Convolution Integral

The obtained expression for the zero-state response y(t) is called the convolution integral.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

The convolution integral is written as:

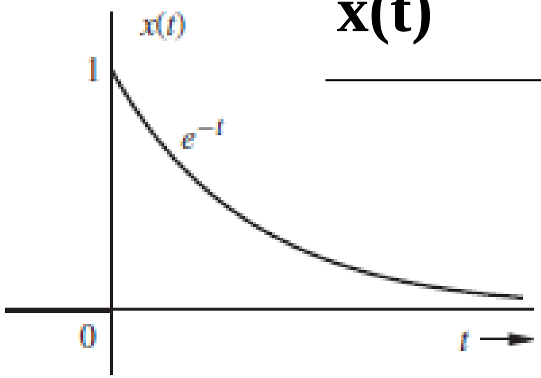
$$y(t) = x(t) * h(t)$$

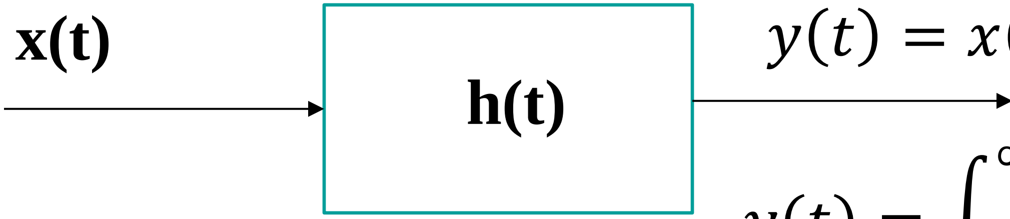


Example:

For an LTIC system with the unit impulse response $h(t) = e^{-2t}u(t)$ determine the response y(t) for the input $x(t) = e^{-t}u(t)$

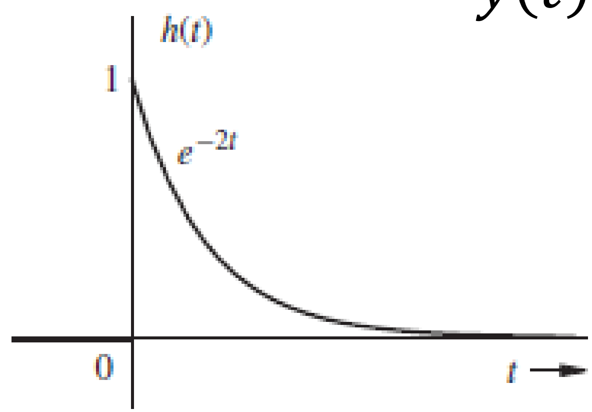
Solution:





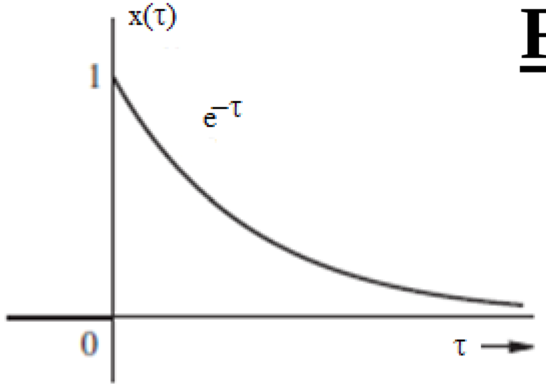
$$y(t) = x(t) * h(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



Solution:

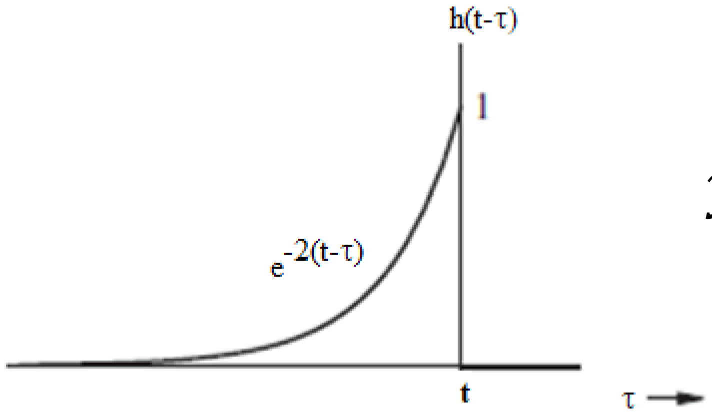
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



For t≥ 0

$$y(t) = \int_0^t e^{-\tau} \times e^{-2(t-\tau)} d\tau$$

$$y(t) = \int_0^t e^{\tau} \times e^{-2t} d\tau$$



$$y(t) = e^{-2t} \int_0^t e^{\tau} d\tau$$

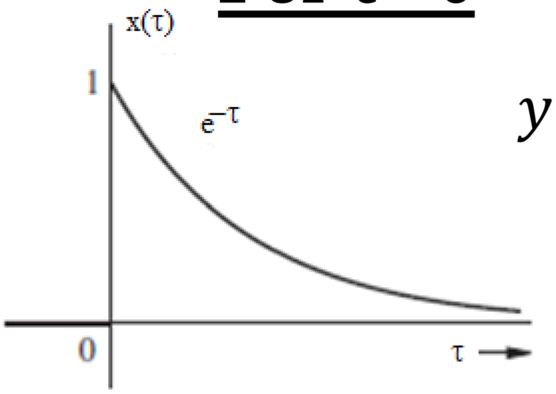
$$y(t) = e^{-2t} [e^{\tau}]_0^t$$

$$y(t) = e^{-2t}(e^t - 1)$$

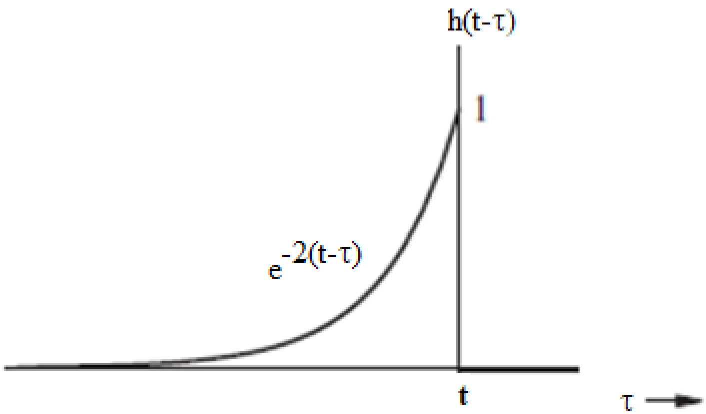
$$y(t) = e^{-t} - e^{-2t}$$

Solution:

For t< 0



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0$$



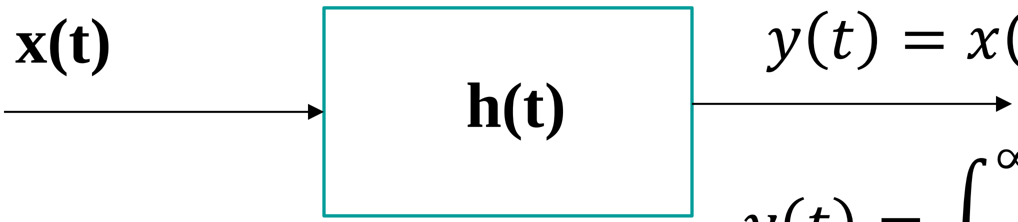
$$y(t) = \begin{cases} e^{-t} - e^{-2t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$y(t) = (e^{-t} - e^{-2t})u(t)$$

Example:

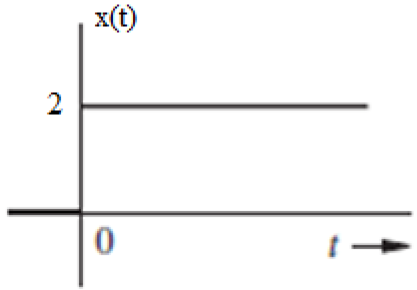
For an **LTIC** system with the unit impulse response $h(t) = 6e^{-t}u(t)$ determine the response y(t) for the input $x(t) = 2u(t)$

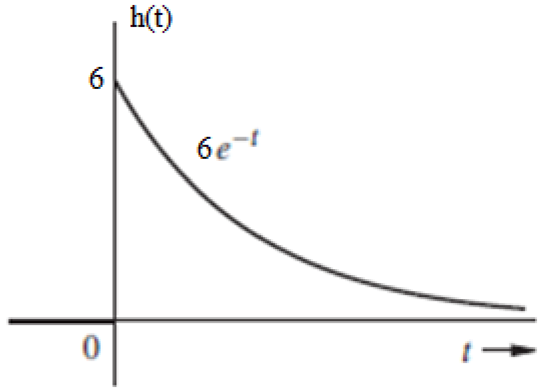
Solution:



$$y(t) = x(t) * h(t)$$

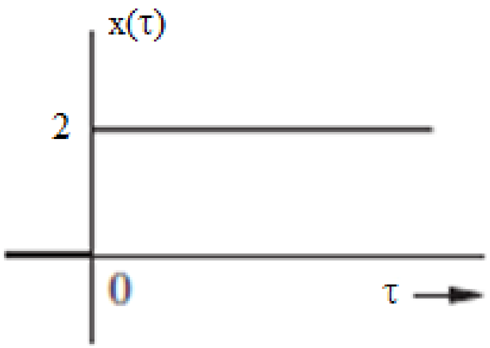
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$





Solution:

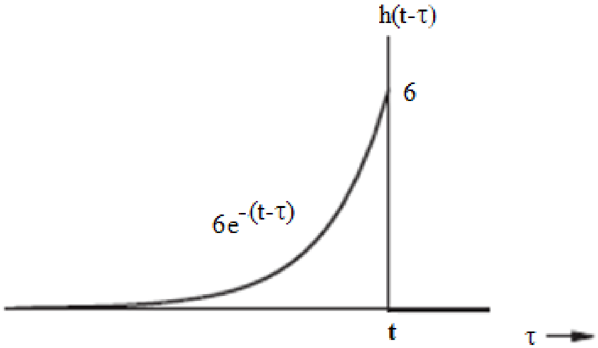
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



For t≥ 0

$$y(t) = \int_{0}^{t} 2 \times 6e^{-(t-\tau)} d\tau$$

$$y(t) = 12\int_{0}^{t} e^{\tau} \times e^{-t} d\tau$$



$$y(t) = 12e^{-t}\int_{0}^{t} e^{\tau} d\tau$$

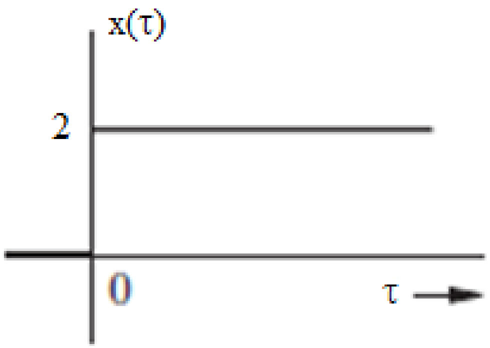
$$y(t) = 12e^{-t}[e^{\tau}]_{0}^{t}$$

$$y(t) = 12e^{-t}(e^{t} - 1)$$

$$y(t) = 12(1 - e^{-t})$$

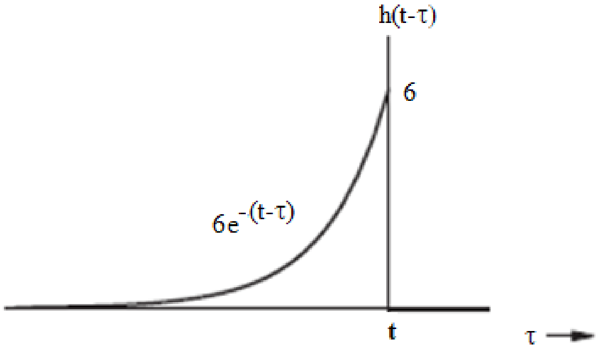
Solution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



For t< 0

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = 0$$

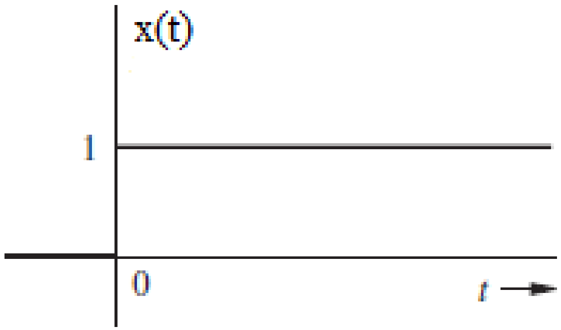


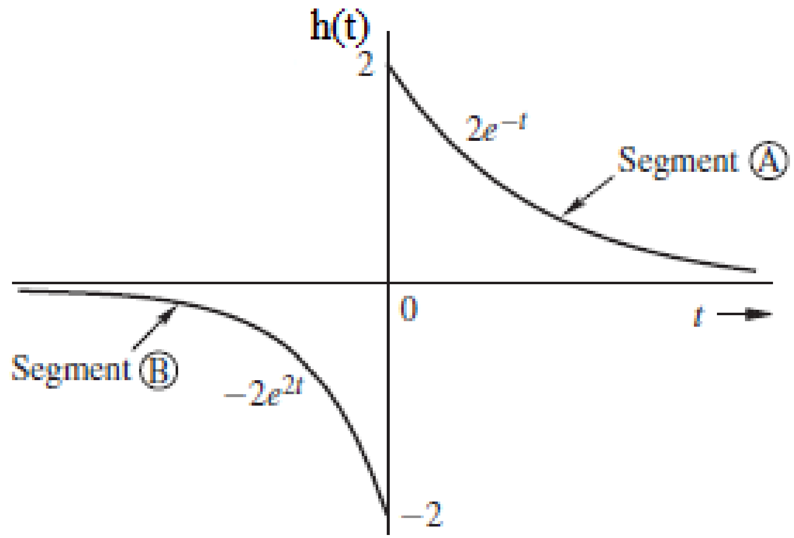
$$y(t) = \begin{cases} 12(1-e^{-t}) & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$y(t) = 12(1-e^{-t})u(t)$$

Example:

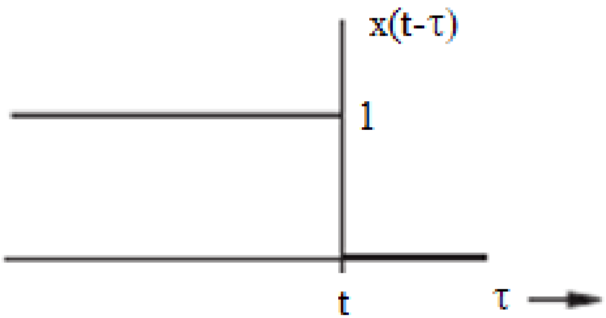
Find y(t) = x(t) ∗ h(t) for the functions shown in figure.

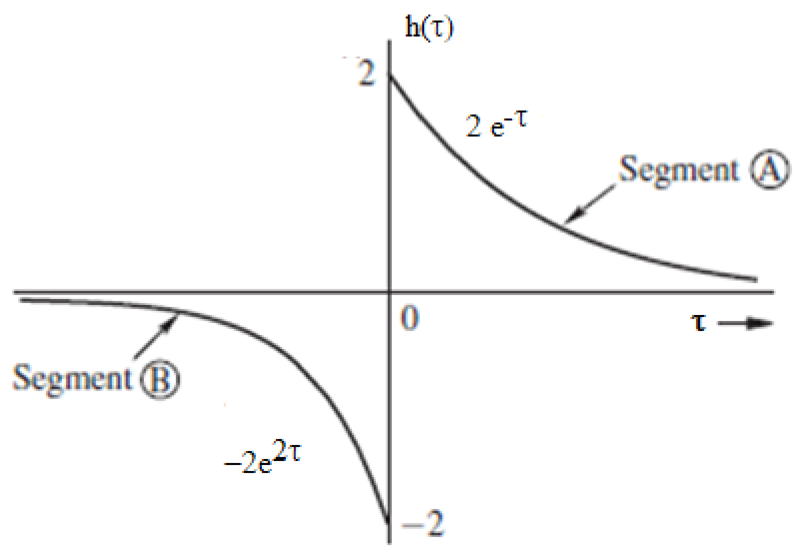




Solution:

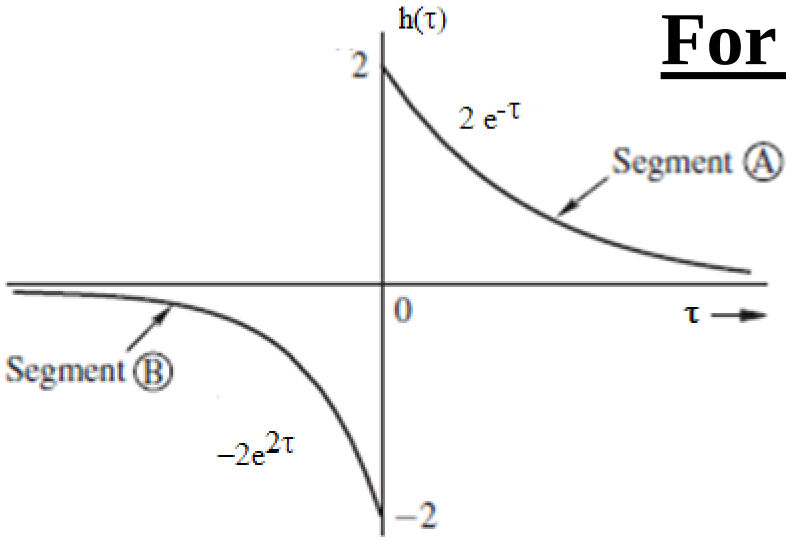
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

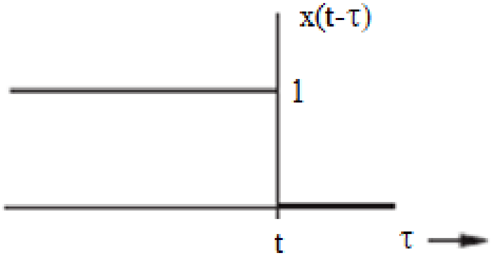




Solution:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$





For t< 0

$$y(t) = \int_{-\infty}^{t} -2e^{2\tau}d\tau$$

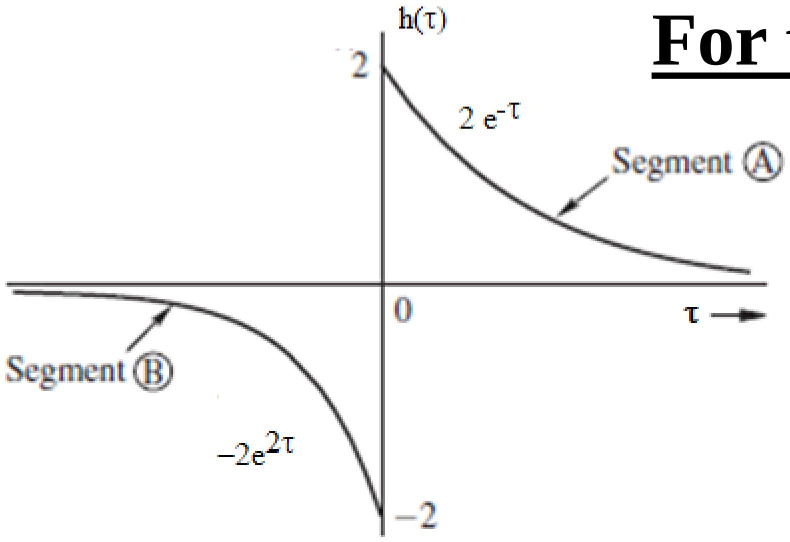
$$y(t) = -2\left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{t}$$

$$y(t) = -2\left[\frac{e^{2t} - e^{-\infty}}{2}\right]_{-\infty}^{t}$$

$$y(t) = -e^{2t}$$

Solution:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

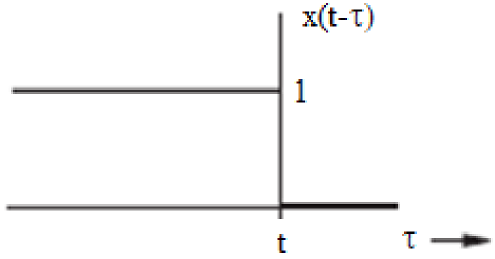


For t ≥ 0

$$y(t) = \int_{-\infty}^{0} -2e^{2\tau}d\tau + \int_{0}^{t} 2e^{-\tau}d\tau$$

$$y(t) = -2\left[\frac{e^{2\tau}}{2}\right]_{-\infty}^{0} + 2\left[\frac{e^{-\tau}}{-1}\right]_{0}^{t}$$

$$y(t) = -2\left[\frac{e^{0} - e^{-\infty}}{2}\right] + 2\left[\frac{e^{-t} - e^{0}}{-1}\right]$$



$$y(t) = -1 - 2(e^{-t} - 1)$$

$$y(t) = 1 - 2e^{-t}$$

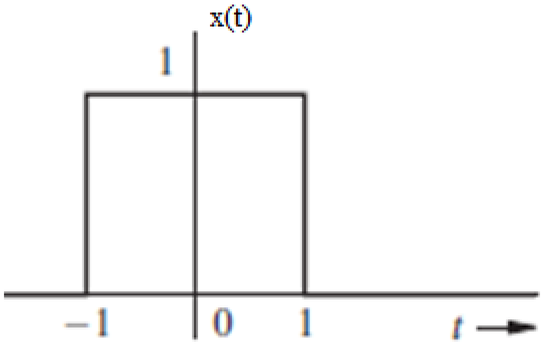
Solution:

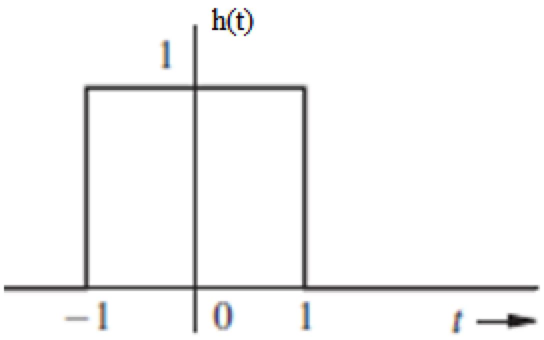
$$y(t) = \begin{cases} 1 - 2e^{-t} & t \geq 0 \\ -e^{2t} & t < 0 \end{cases}$$

$$y(t) = 1 - 2e^{-t}u(t) - e^{2t}u(-t)$$

Example:

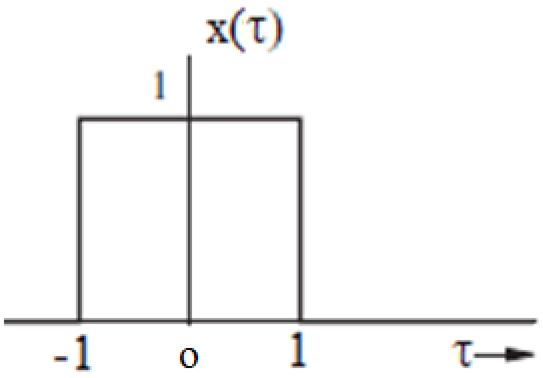
Find $y(t) = x(t) * h(t)$ for the functions shown in figure.

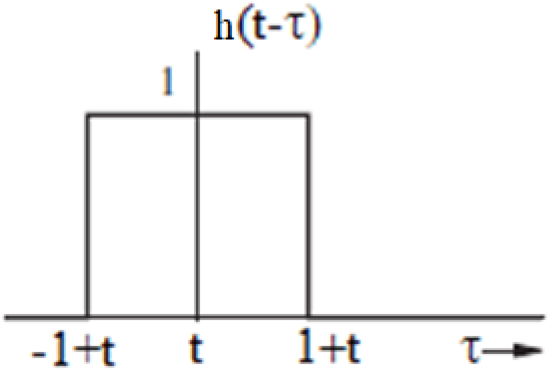




Solution:

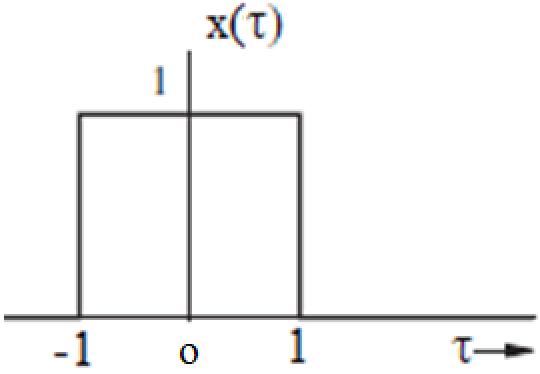
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

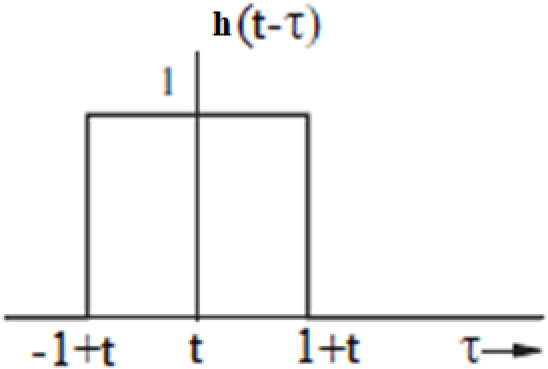




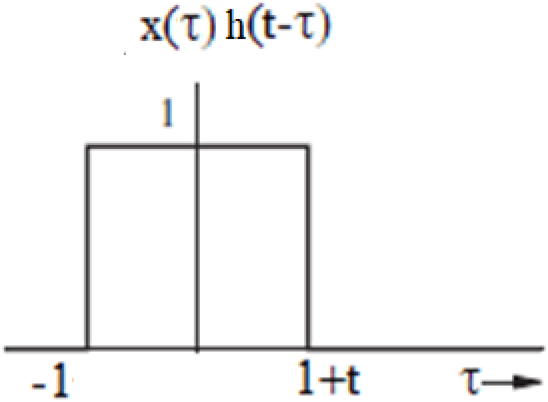
Solution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$





For -2≤ t ≤ 0



$$y(t) = \int_{-1}^{1+t} x(\tau)h(t-\tau)d\tau$$

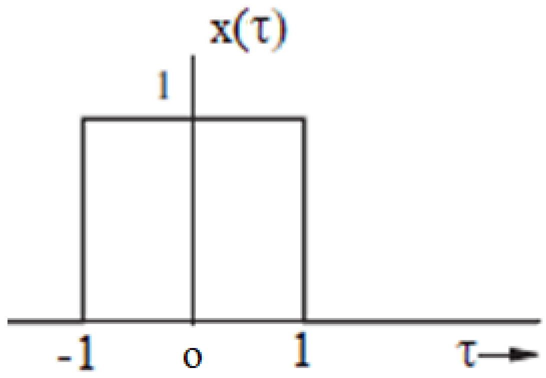
$$y(t) = Area$$

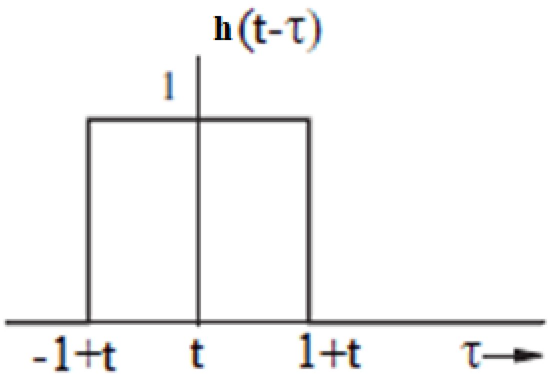
$$y(t) = (t+2) \times 1$$

$$y(t) = t + 2$$

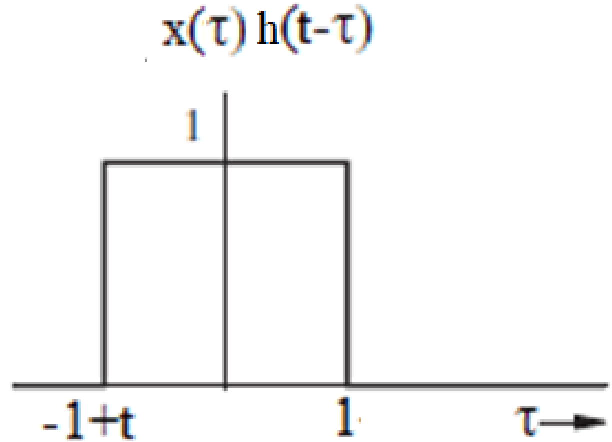
Solution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$





For 0≤ t ≤ 2



$$y(t) = \int_{-1+t}^{1} x(\tau)h(t-\tau)d\tau$$

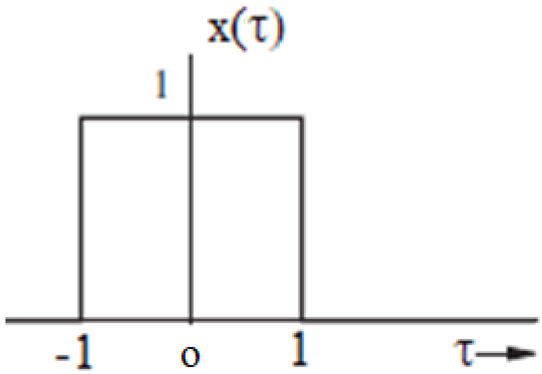
$$y(t) = Area$$

$$y(t) = (2-t) \times 1$$

$$y(t) = 2-t$$

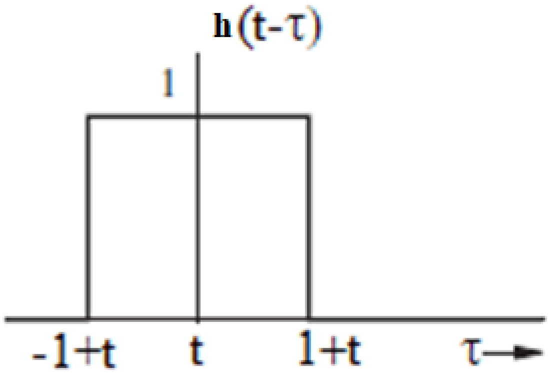
Solution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



For t ≥ 2

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$



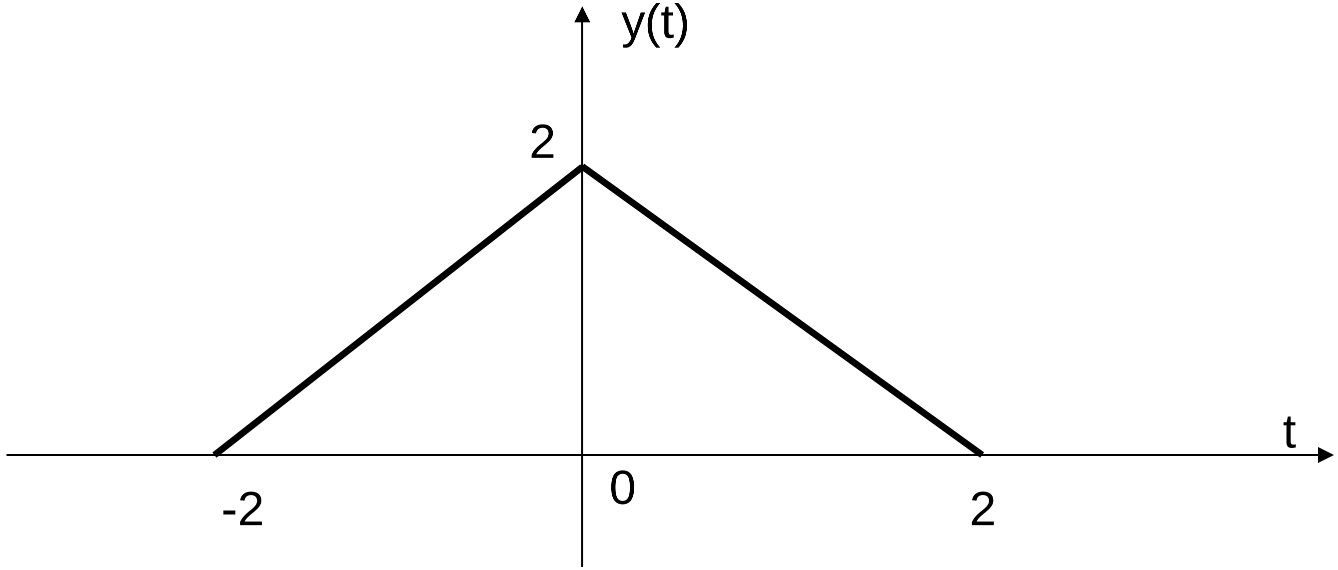
$$y(t) = zero$$

Solution:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

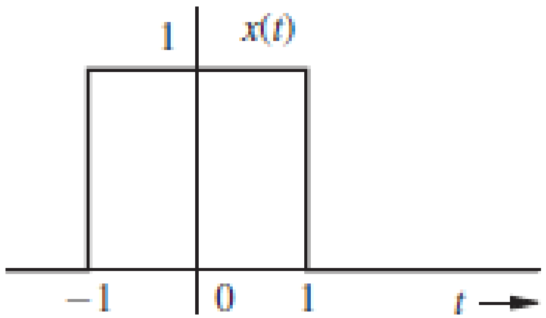
$$y(t) = \begin{cases} 2+t & -2 \le t \le 0 \\ 2-t & 0 \le t \le 2 \\ 0 & Otherwise \end{cases}$$

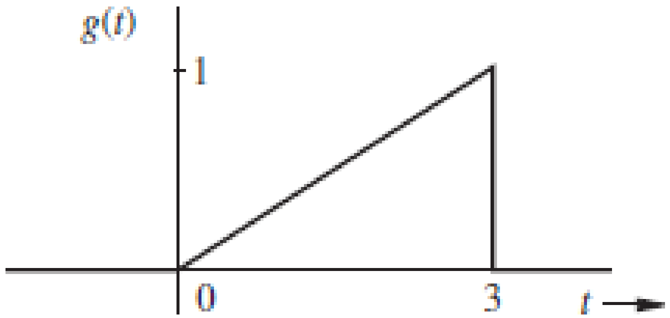
$$y(t) = (2+t)\big(u(t+2) - u(t)\big) + (2-t)\big(u(t) - u(t-2)\big)$$



Example:

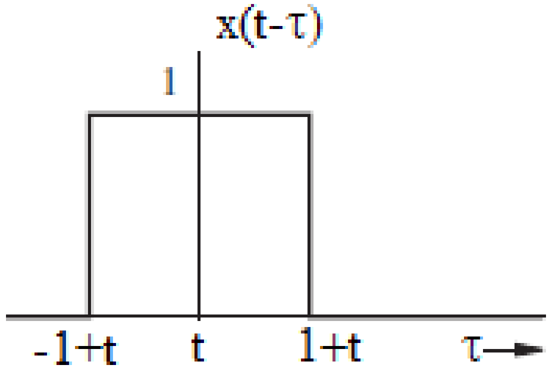
Find $y(t) = x(t) * g(t)$ for the functions shown in figure.

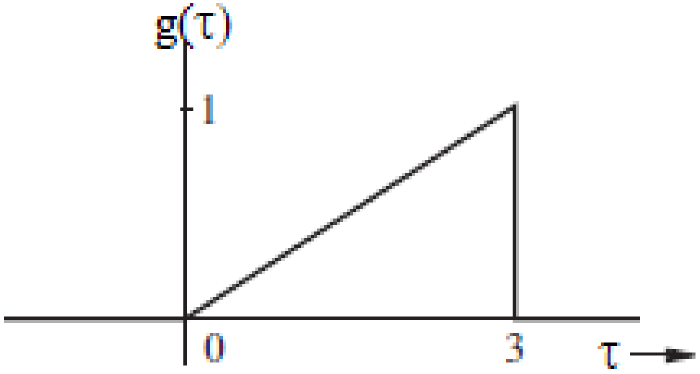




Solution:

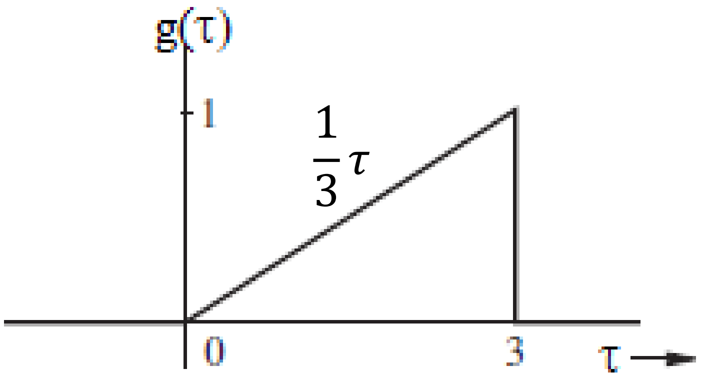
$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$

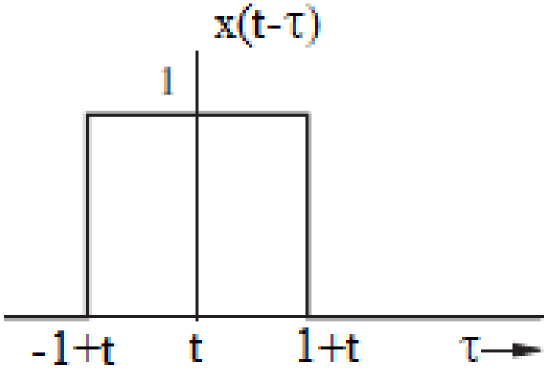




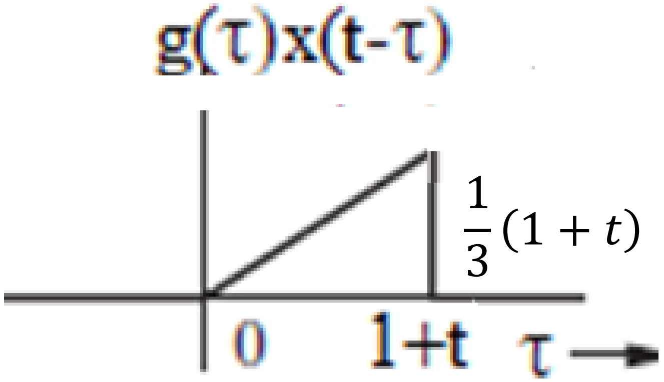
Solution:

$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$





For -1≤ t ≤ 1



$$y(t) = \int_{0}^{1+t} g(\tau)x(t-\tau)d\tau$$

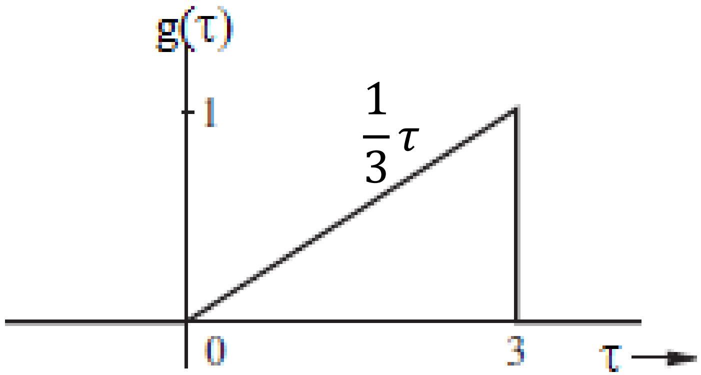
$$y(t) = Area$$

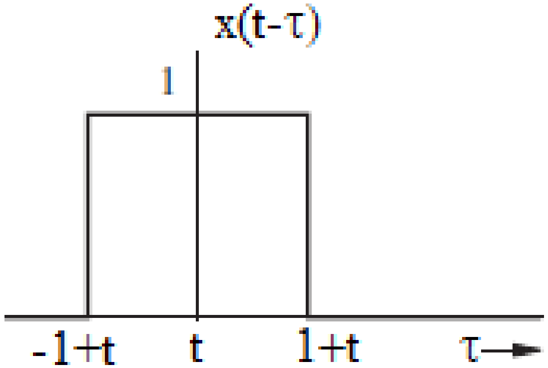
$$y(t) = \frac{1}{2}(1+t) \times \frac{1}{3}(1+t)$$

$$y(t) = \frac{1}{6}(1+t)^2$$

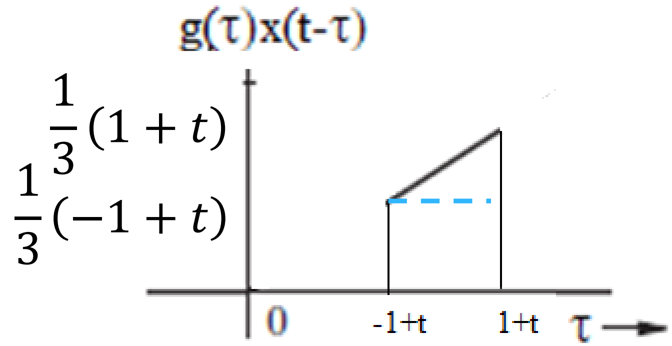
Solution:

$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$





For 1≤ t ≤ 2



$$y(t) = \int_{-1+t}^{1+t} g(\tau)x(t-\tau)d\tau$$

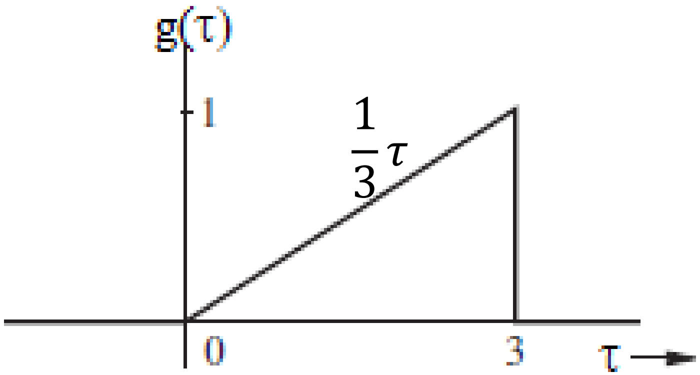
$$y(t) = Area$$

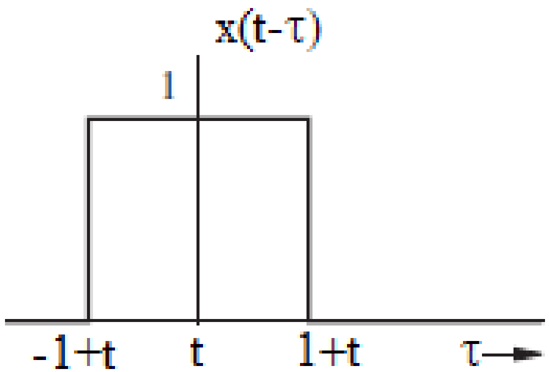
$$y(t) = 2 \times \frac{1}{3}(-1+t) + \frac{1}{2} \times 2 \times \frac{2}{3}$$

$$y(t) = -\frac{2}{3} + \frac{2}{3}t + \frac{2}{3} = \frac{2}{3}t$$

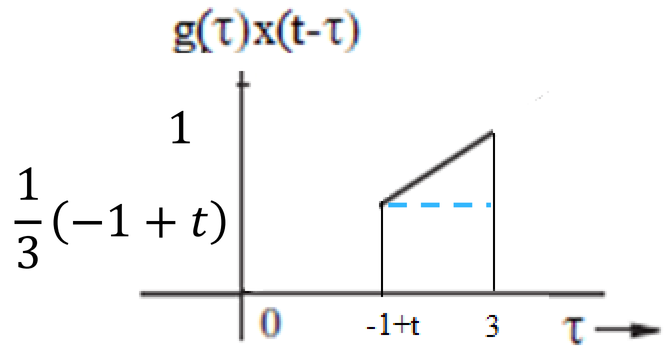
Solution:

$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$





For 2 ≤ t ≤ 4



$$y(t) = \int_{-1+t}^{3} g(\tau)x(t-\tau)d\tau$$

$$y(t) = Area$$

$$y(t) = (4-t) \times \frac{1}{3}(-1+t) + \frac{1}{2} \times (4-t) \times \left(1 - \frac{1}{3}(-1+t)\right)$$

$$y(t) = \frac{(4-t)(-1+t)}{3} + \frac{(4-t)}{2} - \frac{(4-t)(-1+t)}{6}$$

Solution:

$$y(t) = \frac{(4-t)(-1+t)}{3} + \frac{(4-t)}{2} - \frac{(4-t)(-1+t)}{6}$$

$$y(t) = \frac{(4-t)(-1+t)}{6} + \frac{(4-t)}{2}$$

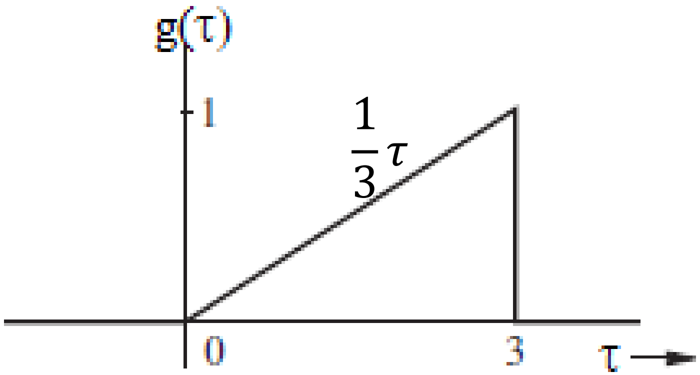
$$y(t) = \frac{1}{6}\big((4-t)(-1+t) + 3(4-t)\big)$$

$$y(t) = \frac{1}{6}(-4 + 4t + t - t^2 + 12 - 3t)$$

$$y(t) = \frac{1}{6}(-t^2 + 2t + 8)$$

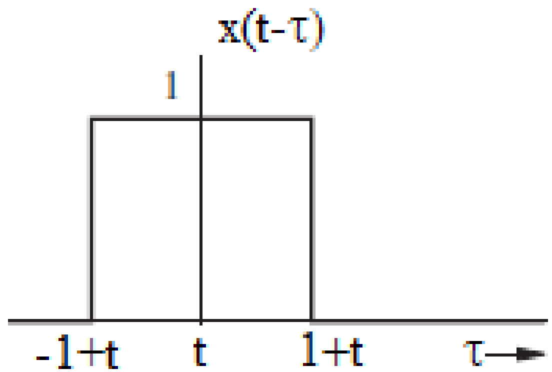
Solution:

$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$



For t ≥ 4

$$y(t) = \int_{-1+t}^{3} g(\tau)x(t-\tau)d\tau = 0$$



Solution:

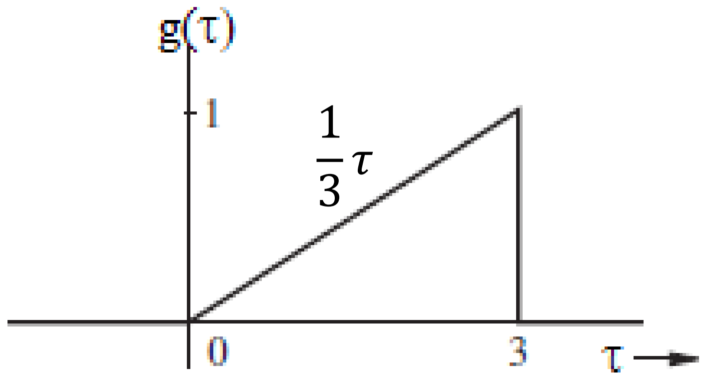
$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$

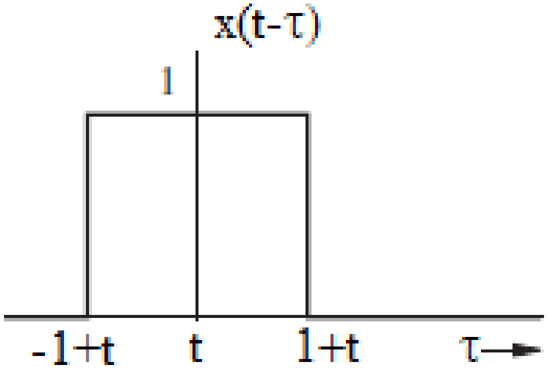
$$y(t) = \begin{cases} \frac{1}{6}(1+t)^2 & -1 \le t \le 1 \\ \frac{2}{3}t & 1 \le t \le 2 \\ \frac{1}{6}(-t^2+2t+8) & 2 \le t \le 4 \\ 0 & Otherwise \end{cases}$$

$$y(t) = \left(\frac{1}{6}(1+t)^2\right)\big(u(t+1)-u(t-1)\big) + \frac{2}{3}t\big(u(t-1)-u(t-2)\big) + \left(\frac{1}{6}(-t^2+2t+8)\right)\big(u(t-1)-u(t-4)\big)$$

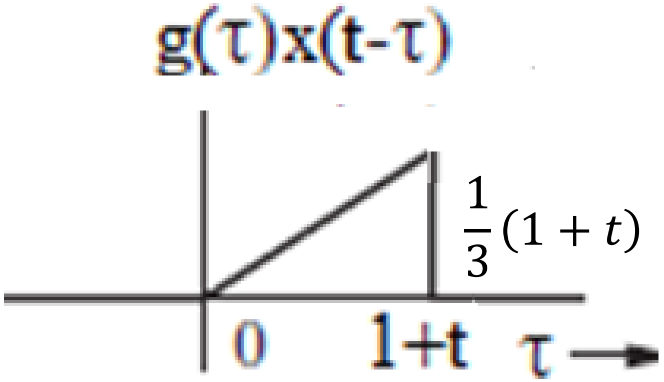
Another Solution:

$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$





For -1≤ t ≤ 1



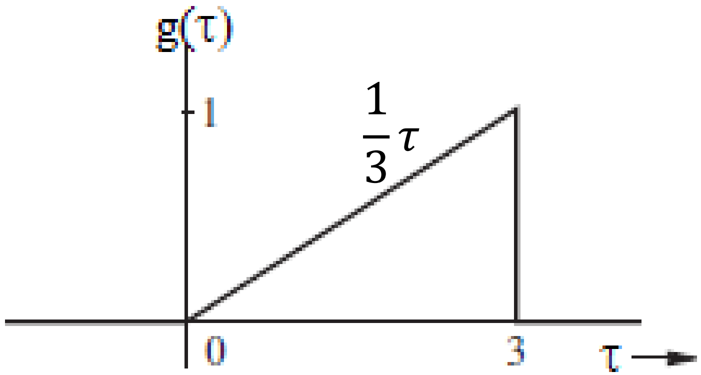
$$y(t) = \int_{0}^{1+t} \frac{1}{3}(\tau)\ d\tau$$

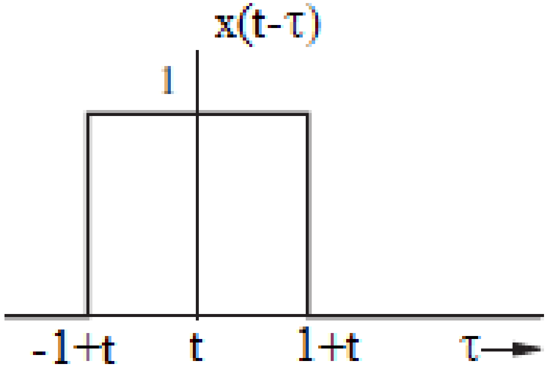
$$y(t) = \left[\frac{\tau^2}{6}\right]_0^{1+t} = \left[\frac{(1+t)^2}{6} - 0\right]$$

$$y(t) = \frac{1}{6}(1+t)^2$$

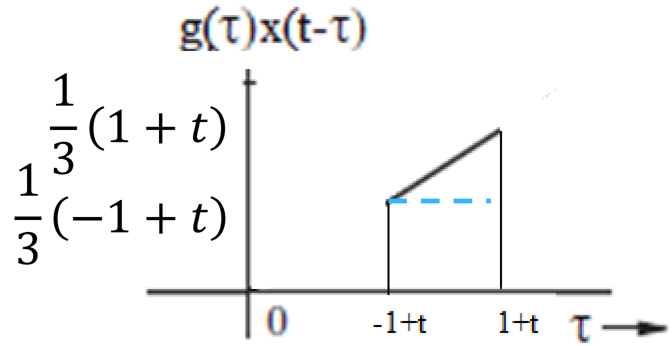
Solution:

$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$





For 1≤ t ≤ 2



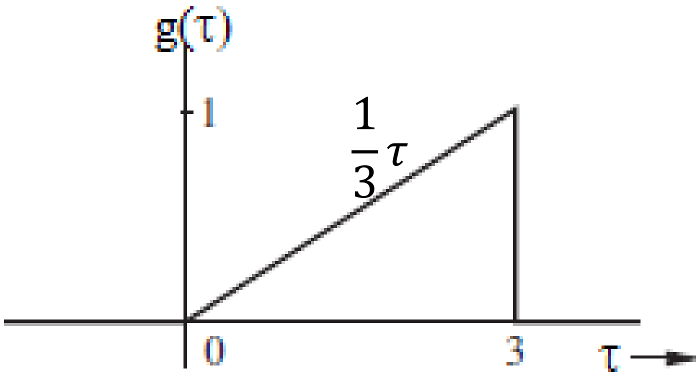
$$y(t) = \int_{-1+t}^{1+t} \frac{1}{3}\tau d\tau$$

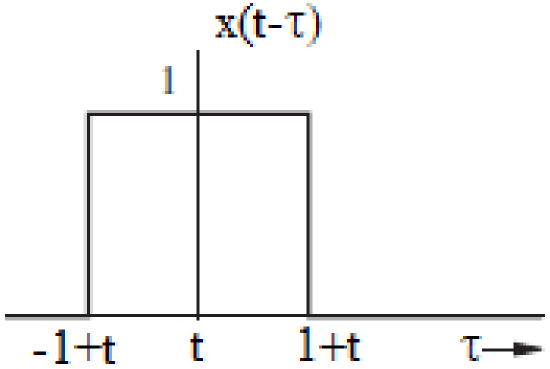
$$y(t) = \left[\frac{\tau^2}{6}\right]_{-1+t}^{1+t} = \left[\frac{(1+t)^2}{6} - \frac{(-1+t)^2}{6}\right]$$

$$y(t) = \frac{2}{3}t$$

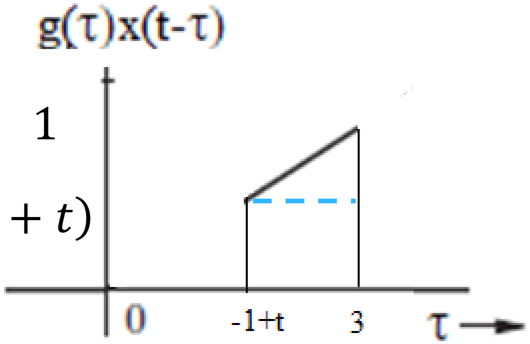
Solution:

$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$





For 2 ≤ t ≤ 4



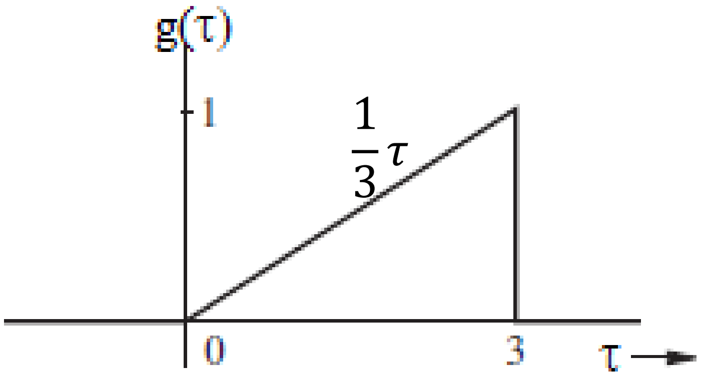
$$y(t) = \int_{-1+t}^{3} \frac{1}{3}\tau d\tau$$

$$y(t) = \left[\frac{\tau^2}{6}\right]_{-1+t}^{3} = \left[\frac{9}{6} - \frac{(-1+t)^2}{6}\right]$$

$$y(t) = \frac{1}{6}(-t^2 + 2t + 8)$$

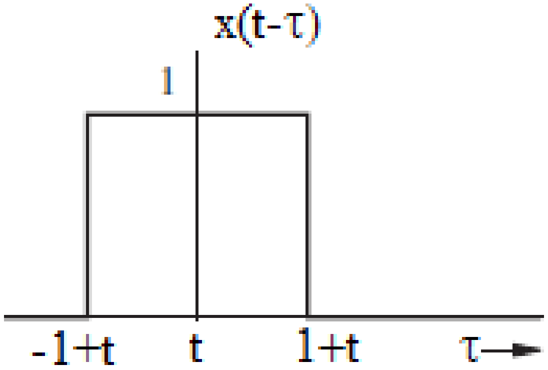
Solution:

$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$



For t ≥ 4

$$y(t) = \int_{-1+t}^{3} g(\tau)x(t-\tau)d\tau = 0$$

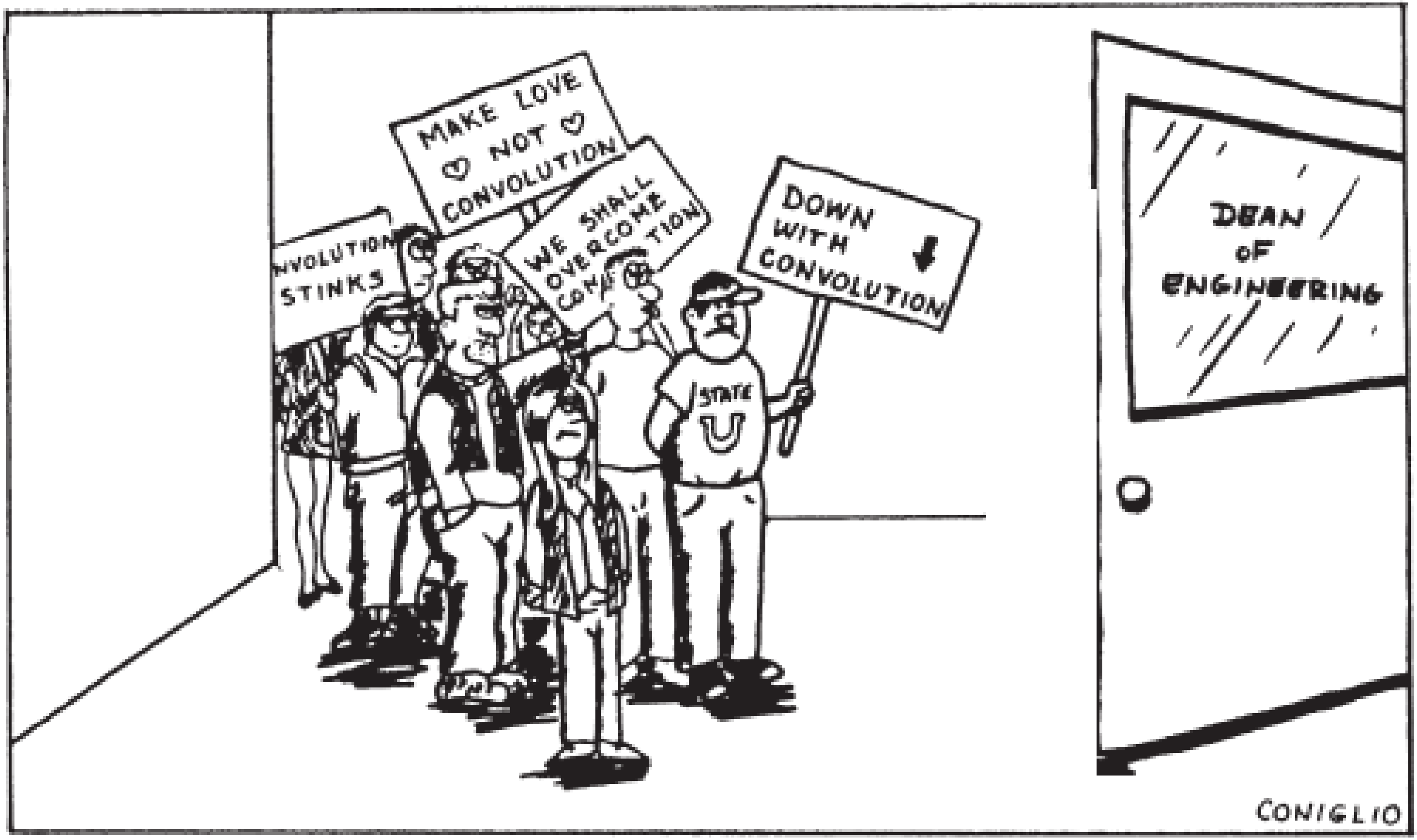


Solution:

$$y(t) = \int_{-\infty}^{\infty} g(\tau)x(t-\tau)d\tau$$

$$y(t) = \begin{cases} \frac{1}{6}(1+t)^2 & -1 \le t \le 1 \\ \frac{2}{3}t & 1 \le t \le 2 \\ \frac{1}{6}(-t^2+2t+8) & 2 \le t \le 4 \\ 0 & Otherwise \end{cases}$$

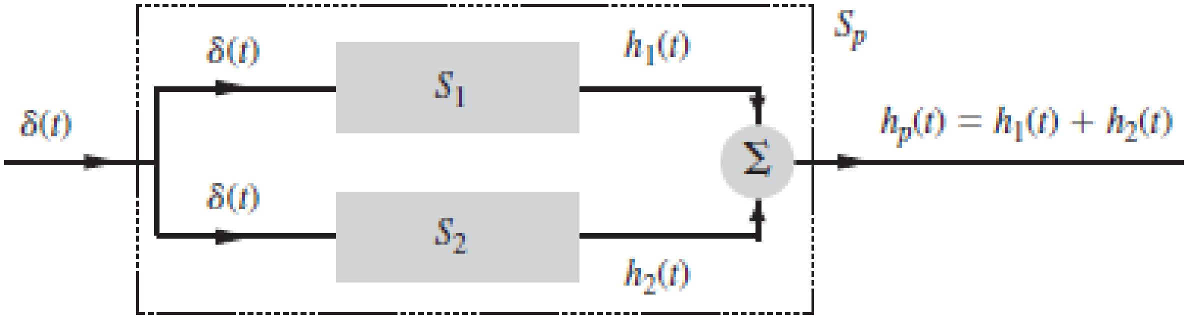
$$y(t) = \left(\frac{1}{6}(1+t)^2\right)\big(u(t+1)-u(t-1)\big)$$
$$+\frac{2}{3}t\big(u(t-1)-u(t-2)\big)$$
$$+\left(\frac{1}{6}(-t^2+2t+8)\right)\big(u(t-1)-u(t-4)\big)$$



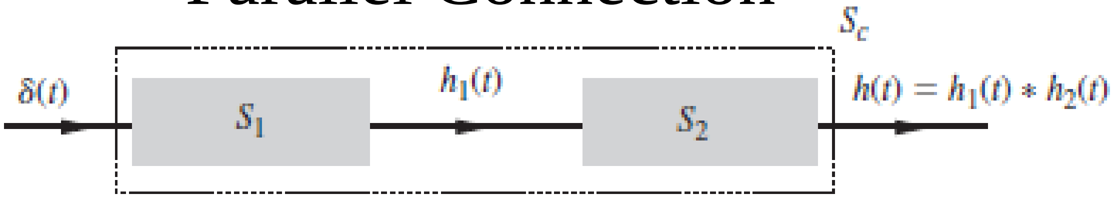
Convolution: Its bark is worse than its bite!

Interconnected systems

- A larger, more complex system can often be viewed as the interconnection of several smaller subsystems, each of which is easier to characterize.
- Knowing the characterizations of these subsystems, it becomes simpler to analyze such large systems.
- We shall consider here two basic interconnections, parallel and cascade.



Parallel Connection



Cascade Connection

Example:

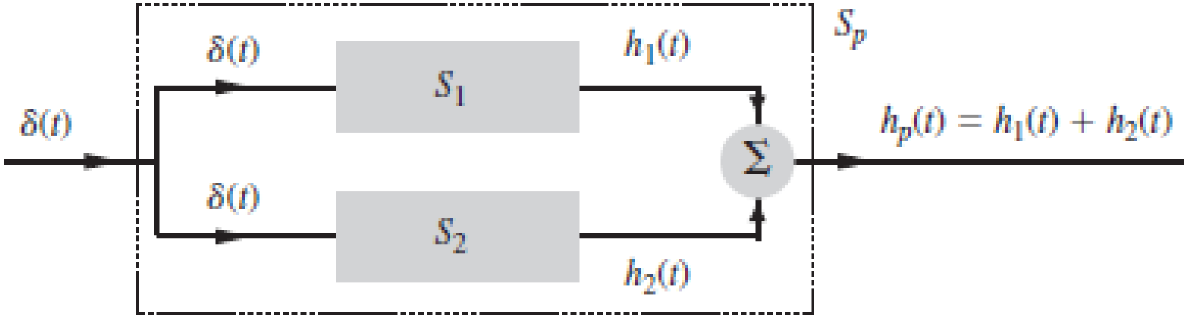
Two LTIC systems have impulse response functions given by $h_1(t)=(1-t)[u(t)-u(t-1)]$ and $h_2(t) = t[u(t+2)-u(t-2)]$.

A. Assume that the two systems are connected in parallel, find the equivalent impulse response function, $h_p(t)$.

B. Assume that the two systems are connected in cascade, find the equivalent impulse response function, $h_s(t)$.

Solution:

A. Parallel Connection:



$$h_1(t) = (1 - t)\big(u(t) - u(t-1)\big)$$

$$h_2(t) = t\big(u(t+2) - u(t-2)\big)$$

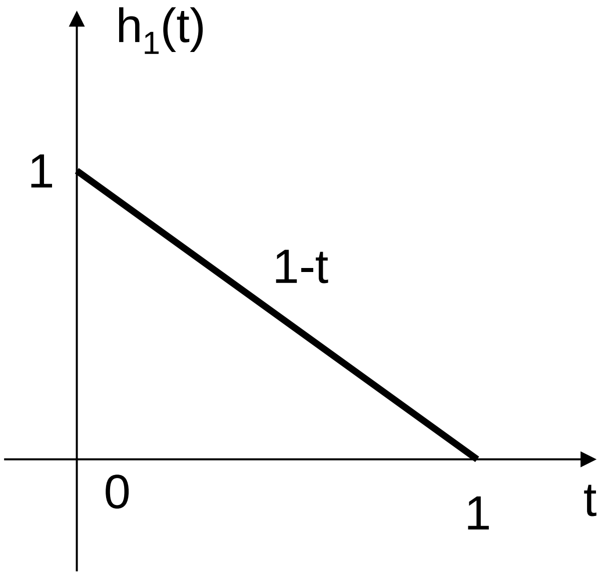
$$h_p(t) = h_1(t) + h_2(t)$$

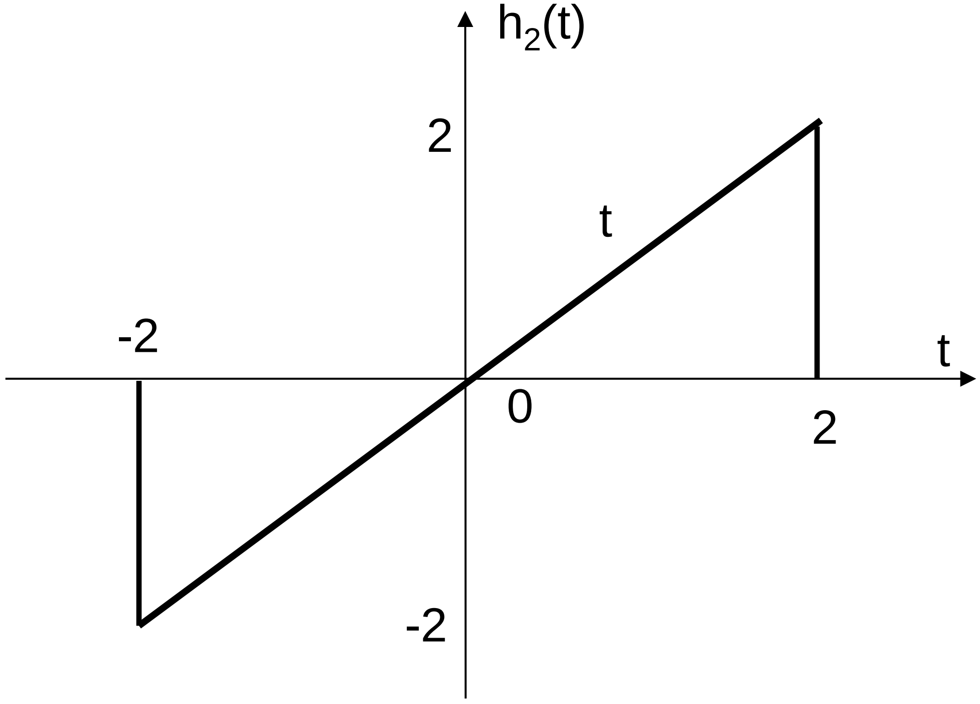
$$h_p(t) = (1 - t)\big(u(t) - u(t-1)\big) + t\big(u(t+2) - u(t-2)\big)$$

Solution:

A. Parallel Connection:

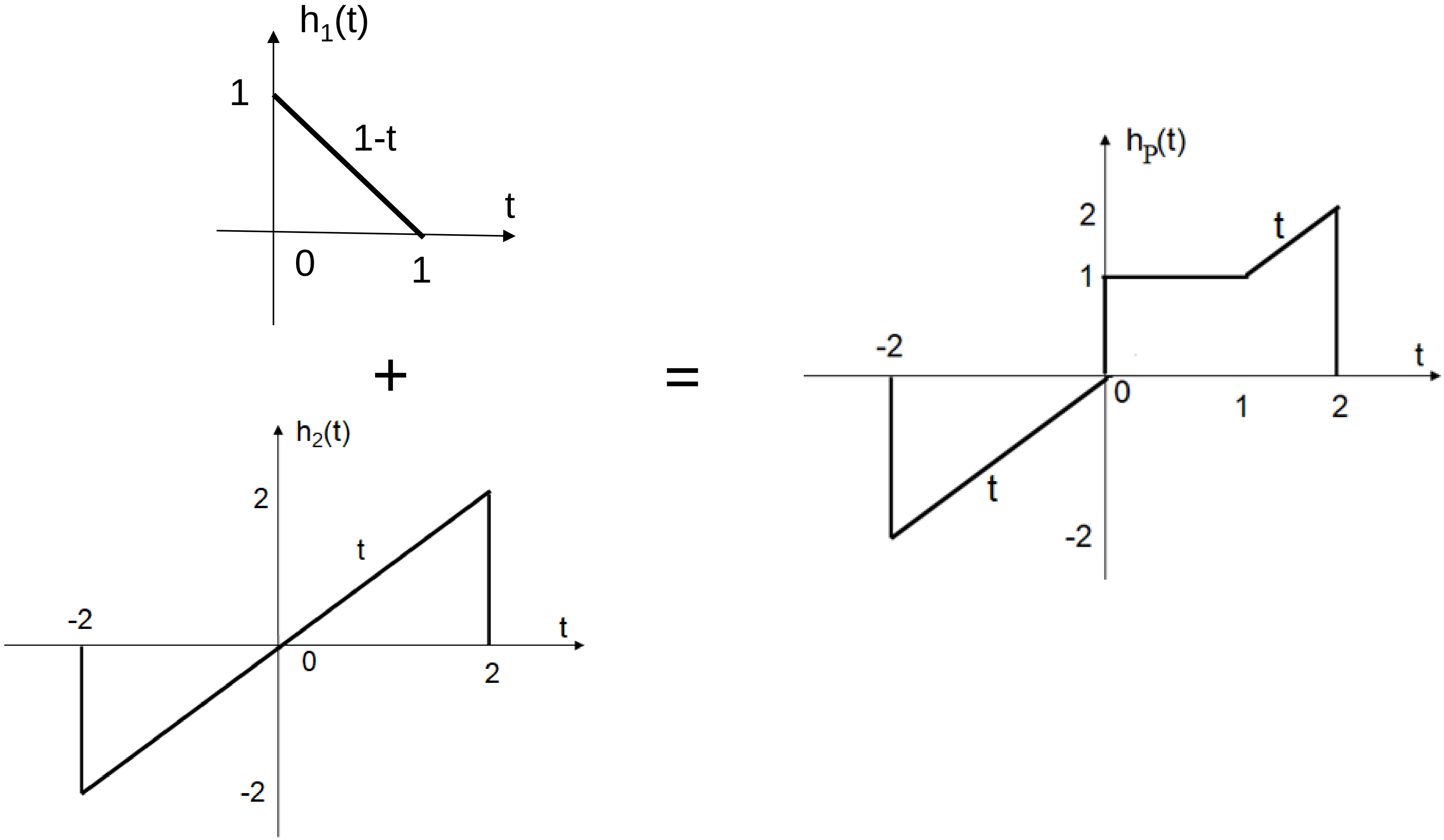
$$h_1(t) = (1 - t)\big(u(t) - u(t-1)\big) \qquad h_2(t) = t\big(u(t+2) - u(t-2)\big)$$





Solution:

A. Parallel Connection:



Solution:

B. Cascade Connection:

$$h_1(t) = (1 - t)\big(u(t) - u(t-1)\big)$$

$$h_2(t) = t\big(u(t+2) - u(t-2)\big)$$

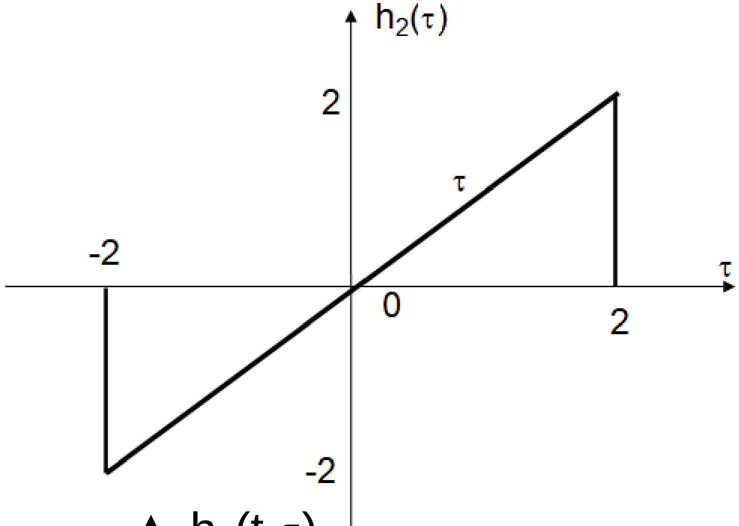
$$h_s(t) = h_1(t) * h_2(t)$$

$$h_s(t) = \int_{-\infty}^{\infty} h_2(\tau) h_1(t-\tau)\, d\tau$$

Solution:

B. Cascade Connection:

$$h_s(t) = \int_{-\infty}^{\infty} h_2(\tau) h_1(t-\tau) d\tau$$

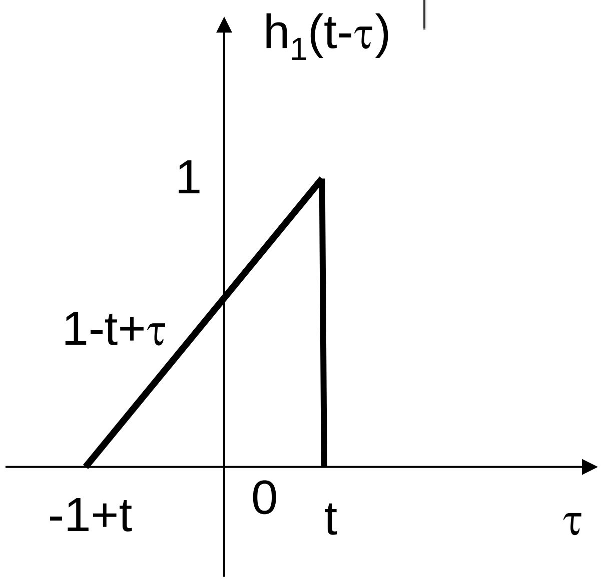


For -2≤ t ≤ -1

$$h_s(t) = \int_{-2}^{t} \tau(1-t+\tau) d\tau$$

$$h_s(t) = \int_{-2}^{t} \tau(1-t) + \tau^2 d\tau$$

$$h_s(t) = \left[(1-t)\frac{\tau^2}{2} + \frac{\tau^3}{3}\right]_{-2}^{t}$$

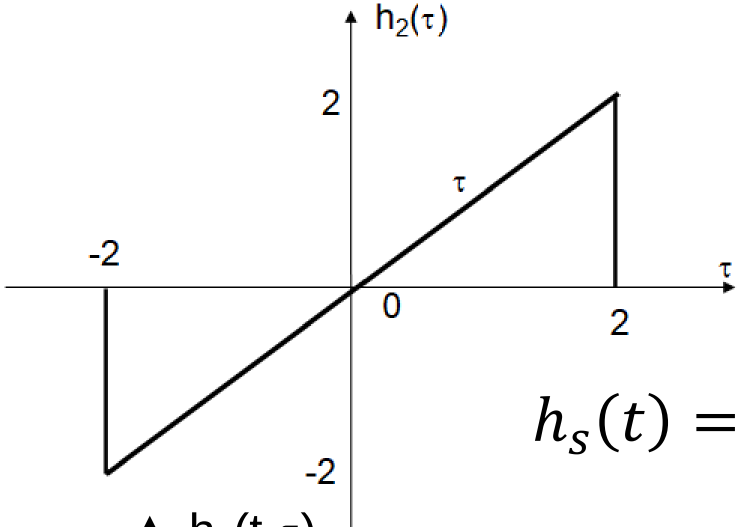


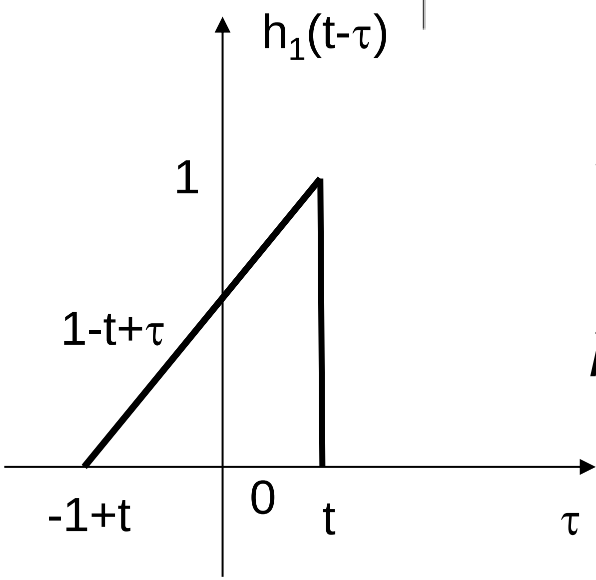
$$h_s(t) = (1-t)\frac{t^2}{2} + \frac{t^3}{3} - \left(2(1-t) - \frac{8}{3}\right)$$

Solution:

B. Cascade Connection:

$$h_s(t) = \int_{-\infty}^{\infty} h_2(\tau) h_1(t-\tau) d\tau$$





For -2≤ t ≤ -1

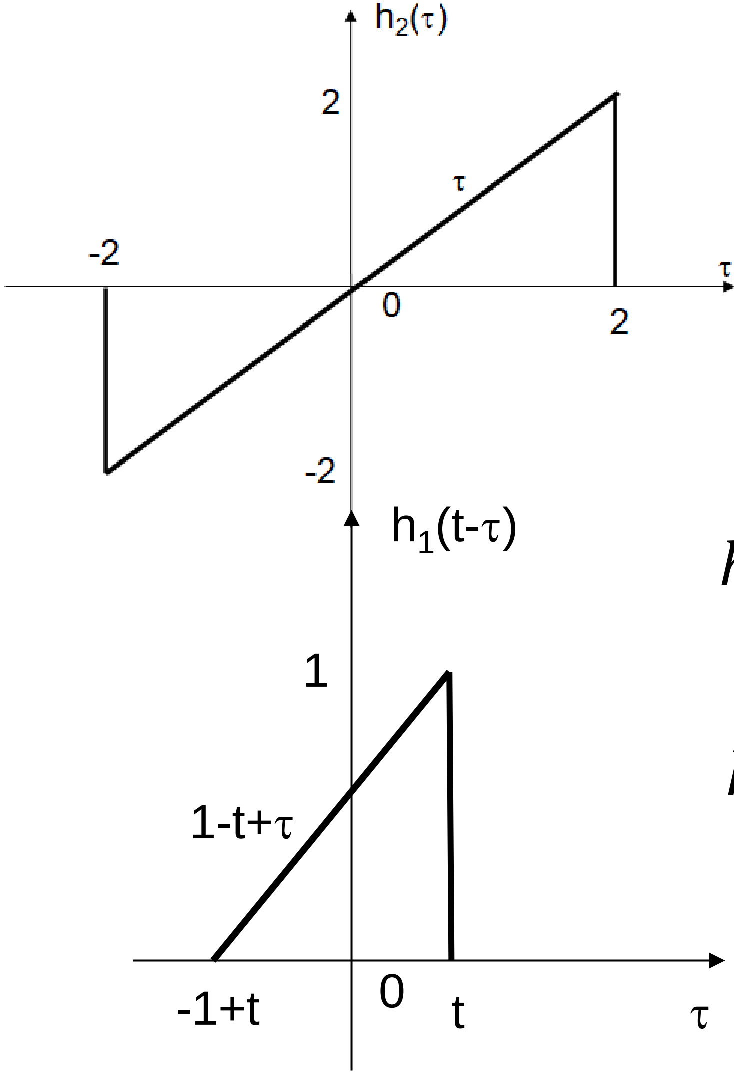
$$h_s(t) = (1-t)\frac{t^2}{2} + \frac{t^3}{3} - \left(2(1-t) - \frac{8}{3}\right)$$

$$h_s(t) = \frac{t^2}{2} - \frac{t^3}{2} + \frac{t^3}{3} - 2 + 2t + \frac{8}{3}$$

$$h_s(t) = -\frac{t^3}{6} + \frac{t^2}{2} + 2t + \frac{2}{3}$$

Solution:

B. Cascade Connection:



$$h_s(t) = \int_{-\infty}^{\infty} h_2(\tau) h_1(t-\tau) d\tau$$

For -1 ≤ t ≤ 2

$$h_s(t) = \int_{-1+t}^{t} \tau(1-t+\tau) d\tau$$

$$h_s(t) = \int_{-1+t}^{t} \tau(1-t) + \tau^2 d\tau$$

$$h_s(t) = \left[(1-t)\frac{\tau^2}{2} + \frac{\tau^3}{3} \right]_{-1+t}^{t}$$

Solution:

B. Cascade Connection:

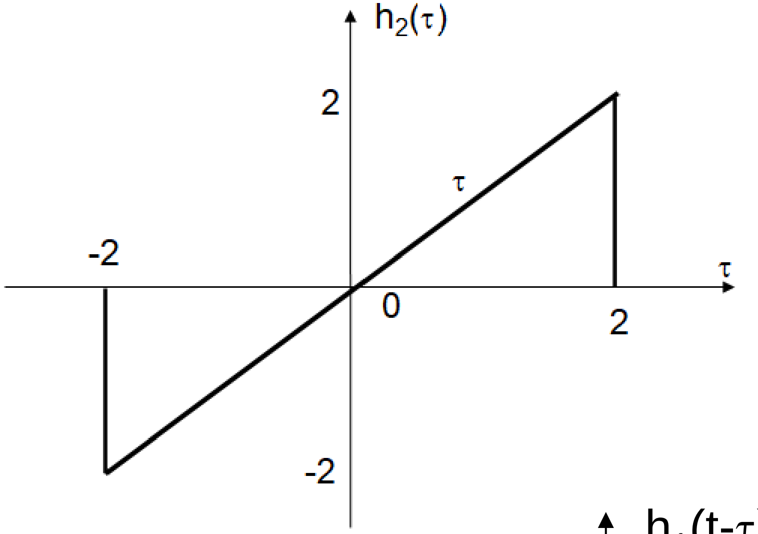
$$h_s(t) = \int_{-\infty}^{\infty} h_2(\tau) h_1(t-\tau) d\tau$$

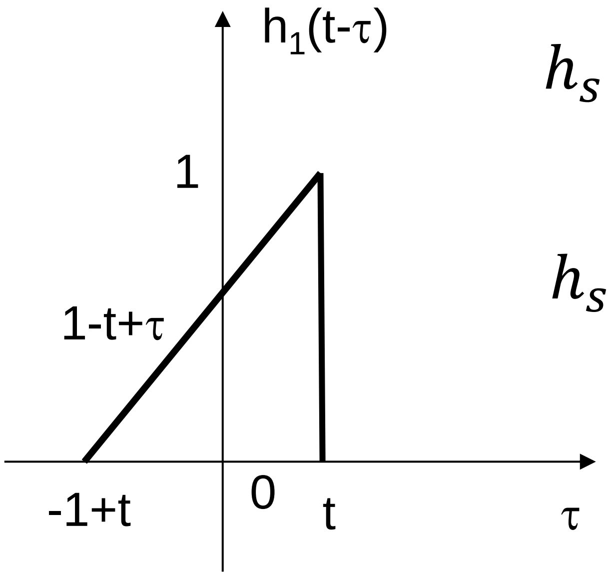
For -1 ≤ t ≤ 2

$$h_s(t) = (1-t)\frac{t^2}{2} + \frac{t^3}{3} - \left((1-t)\frac{(-1+t)^2}{2} + \frac{(-1+t)^3}{3}\right)$$

Solution:

B. Cascade Connection:





$$h_s(t) = \int_{-\infty}^{\infty} h_2(\tau) h_1(t-\tau)\,d\tau$$

For $2 \leq t \leq 3$

$$h_s(t) = \int_{-1+t}^{2} \tau(1-t+\tau)\,d\tau$$

$$h_s(t) = \int_{-1+t}^{2} \tau(1-t) + \tau^2\,d\tau$$

$$h_s(t) = \left[(1-t)\frac{\tau^2}{2} + \frac{\tau^3}{3}\right]_{-1+t}^{2}$$

Solution:

B. Cascade Connection:

$$h_s(t) = \int_{-\infty}^{\infty} h_2(\tau)h_1(t-\tau)d\tau$$

For 2 ≤ t ≤ 3

$$h_s(t) = \left[(1-t)\frac{\tau^2}{2} + \frac{\tau^3}{3}\right]_{-1+t}^{2}$$

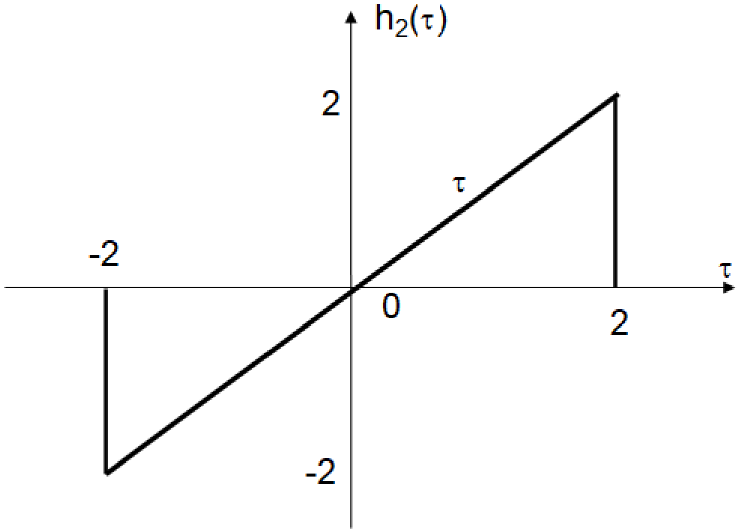
$$h_s(t) = 2 + \frac{8}{3} - \left((1-t)\frac{(-1+t)^2}{2} + \frac{(-1+t)^2}{3}\right)$$

$$h_s(t) = \frac{14}{3} - \left((1-t)\frac{(-1+t)^2}{2} + \frac{(-1+t)^2}{3}\right)$$

Solution:

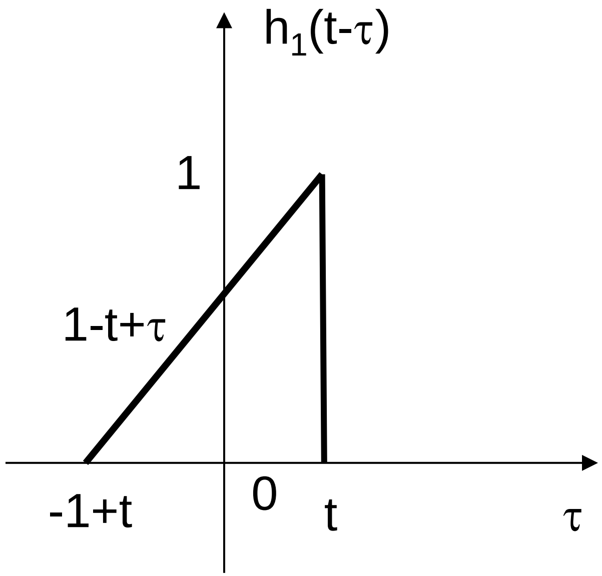
B. Cascade Connection:

$$h_s(t) = \int_{-\infty}^{\infty} h_2(\tau) h_1(t-\tau) d\tau$$



For t ≥ 3

$$h_s(t) = 0$$



Solution:

B. Cascade Connection:

$$h_s(t) = \int_{-\infty}^{\infty} h_2(\tau) h_1(t-\tau)\, d\tau$$

$$y(t) = \begin{cases} -\dfrac{t^3}{6} + \dfrac{t^2}{2} + 2t + \dfrac{2}{3} & -2 \le t \le -1 \\ (1-t)\dfrac{t^2}{2} + \dfrac{t^3}{3} - \left((1-t)\dfrac{(-1+t)^2}{2} + \dfrac{(-1+t)^3}{3} \right) & -1 \le t \le 2 \\ \dfrac{14}{3} - \left((1-t)\dfrac{(-1+t)^2}{2} + \dfrac{(-1+t)^2}{3} \right) & 2 \le t \le 3 \\ 0 & Otherwise \end{cases}$$