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Calculus 3 HW.2

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Q1 $\vec{a}: \langle 2,-6,2 \rangle$ $\vec{b}: \langle 0,4,-2 \rangle$ $\vec{c}: \langle 2,2,-4 \rangle$

I. $\vec{a} \times \vec{b} =$

$$\begin{array}{ccc} + & - & + \\ \left[\begin{array}{ccc} \hat{i} & \hat{j} & k \\ 2 & -6 & 2 \\ 0 & 4 & -2 \end{array}\right] \end{array} = (12-8)\hat{i} - (-4-0)\hat{j} + (8-0)\hat{k}$$

$\hookrightarrow 4\hat{i} + 4\hat{j} + 8\hat{k}$ $\rightarrow$ $\vec{a} \times \vec{b} = \langle 4, 4, 8 \rangle$

II. $\vec{v}$ & length $= 2$. Orthogonal to $\vec{a}$ and $\vec{b}$

cross product

$\therefore$ from (I) let $\vec{v} = \vec{a} \times \vec{b} = \langle 4, 4, 8 \rangle$ Norm $(|v|) = \sqrt{(4)^2 + (4)^2 + (8)^2} = \sqrt{96} = 4\sqrt{6}$

Unit vector $\rightarrow$ $\hat{v} = \left\langle \frac{4}{4\sqrt{6}}, \frac{4}{4\sqrt{6}}, \frac{8}{4\sqrt{6}} \right\rangle \rightarrow \hat{v} = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle \xrightarrow{\times 2 \text{ length}} \left\langle \frac{2}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{4}{\sqrt{6}} \right\rangle$

III. $\vec{a} \times (\vec{b} \times \vec{c}) \rightarrow \vec{b} \times \vec{c} =$

$$\left[\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{array}\right] = (-16--4)\hat{i} - (0--4)\hat{j} + (0-8)\hat{k} \rightarrow -12\hat{i} - 4\hat{j} - 8\hat{k} \rightarrow \langle -12, -4, -8 \rangle$$

$\hookrightarrow \vec{a} \times (\vec{b} \times \vec{c}) =$

$\langle -12, -4, -8 \rangle$

$$\left[\begin{array}{ccc} \hat{i} & \hat{j} & \hat{k} \\ 2 & -6 & 2 \\ -12 & -4 & -8 \end{array}\right] = (48--8)\hat{i} - (-16--24)\hat{j} + (-8-72)\hat{k}$$

$\hookrightarrow 56\hat{i} - 8\hat{j} - 80\hat{k} \rightarrow \langle 56, -8, -80 \rangle$

Or 2nd way to solve it by using Triple cross product properties.

$$\boxed{\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}}$$

$\hookrightarrow (4 + -12 + -8)\vec{b} - (0 + -24 + -4)\vec{c} = -16\vec{b} + 28\vec{c}$

$\hookrightarrow -16\langle 0,4,-2 \rangle + 28 \langle 2,2,-4 \rangle = \langle 0, -64, 32 \rangle + \langle 56, 56, -112 \rangle$

$\hookrightarrow \langle 56, -8, -80 \rangle$

IV. $proj_{\vec{a}}\vec{c} = \dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|^2} * \vec{a} \longrightarrow \dfrac{4+-12+-8}{\left(\sqrt{4+36+4}\right)^2} * \langle 2,-6,2\rangle = \dfrac{\overset{-4}{\cancel{-16}}}{\underset{11}{\cancel{44}}} * \langle 2,-6,2\rangle \longrightarrow \boxed{\langle \frac{-8}{11}, \frac{24}{11}, \frac{-8}{11}\rangle}$

V. Volume of parallelepiped determined by $\vec{a}, \vec{b}$ and $\vec{c}$.

cross product properties

$|\vec{a}\cdot\vec{b}\times\vec{c}|$

$$\therefore |\vec{a}\cdot\vec{b}\times\vec{c}| = \begin{array}{ccc} + & - & + \\ \end{array}\begin{bmatrix} 2 & -6 & 2 \\ 0 & 4 & -2 \\ 2 & 2 & -4 \end{bmatrix} = |2(-16--4)+6(0--4)+2(0-8)| \rightarrow |2*-12 + 6*4 + 2*-8|$$

$$\rightarrow |-24+24-16| = |-16| = \boxed{16}$$

Q2. $\vec{a} = 2\hat{i}+3\hat{j}-\hat{k}$ $\quad\vec{b} = -3\hat{i}+2\hat{k}$ $\quad\vec{c} = \hat{i}+2\hat{j}-2\hat{k}$ $\quad$ Find:

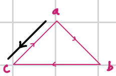
I. $\vec{a}\cdot(\vec{b}\times\vec{c}) = \underset{1st}{\bullet}\ \vec{b}\times\vec{c} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 0 & 2 \\ 1 & 2 & -2 \end{bmatrix} = -4\hat{i} - 4\hat{j} - 6\hat{k} \xrightarrow[\text{then}]{\cdot\vec{a}} \overset{\vec{a}}{\langle 2,3,-1\rangle}\cdot\overset{(\vec{b}\times\vec{c})}{\langle -4,-4,-6\rangle} = -8+-12+6 = \boxed{-14}$

II. $2\vec{b}\times 3\vec{c}\cdot 2\vec{a}$

• priority for cross product so its the same as I. $\vec{a}\cdot(\vec{b}\times\vec{c})$ but multiplied by $\overset{2*3*2}{12}$ $\therefore$ $12 * -14 = \boxed{-168}$

III. $\vec{b}\cdot(\vec{a}\times\vec{c})$

* we know $\vec{a}\cdot(\vec{b}\times\vec{c}) = -14$ in I.



$\therefore$ $\vec{b}\cdot(\vec{a}\times\vec{c}) = -(-14) = \boxed{14}$

IV. $\vec{c}\cdot\vec{b}\times\vec{c}$

$\vec{b}\times\vec{c}$ is perpendicular on $\vec{c}$ → cross product creates a vector that is orthogonal to both vectors $(\vec{b},\vec{c})$

$\therefore$ so the dot product between the vector $\vec{c}$ and the $(\vec{c}\times\vec{b})$ will be zero $\quad\boxed{\vec{c}\cdot\vec{b}\times\vec{c} = \text{zero}}$

Q3. let $\vec{u} = 8\hat{i} + 2\hat{j}$ $\quad \vec{v} = \hat{i} + \hat{j}$ $\quad \vec{w} = \hat{i} - \hat{j}$ $\quad$ find scalar a and b such that $\vec{u} = a\vec{v} + b\vec{w}$

$u = a\vec{v} + b\vec{w} \rightarrow a\hat{i} + a\hat{j} + b\hat{i} - b\hat{j} = 8\hat{i} + 2\hat{j} \rightarrow \hat{i}(a+b) + \hat{j}(a-b) = 8\hat{i} + 2\hat{j}$ $\quad \therefore a+b=8$ —① $\quad a-b=2$ —② $\quad$ ①+②$= a+\cancel{b}+a-\cancel{b} = 8+2 \rightarrow 2a = 10 \quad \therefore a=5 \xrightarrow{\text{in ①}} 5+b=8 \quad \therefore b=3$

Q4 let $\vec{a}, \vec{b}$ non zero vectors such that $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$. Show that they are perpendicular.

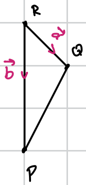
* square both side & equation $|\vec{a}+\vec{b}|^2 = |\vec{a}-\vec{b}|^2 \rightarrow |\vec{a}|^2 + 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 = |\vec{a}|^2 - 2\vec{a}\cdot\vec{b} + |\vec{b}|^2 \rightarrow \frac{4\vec{a}\cdot\vec{b}}{4} = \frac{0}{4} \rightarrow \vec{a}\cdot\vec{b} = 0$

∴ If the dot product is zero, it's either → ~~zero vectors~~ rejected

or → They're perpendicular ※

Q5 find the area of the △ whose vertices are P(1,−1,2) Q(2,1,2) R(1,2,3).

. The Area of △ is one of the cross product applications. Area $= \frac{1}{2}|\vec{a} \times \vec{b}|$



let $\vec{a} = \vec{RQ} = \langle 1, -1, -1 \rangle \rightarrow$ Area $= \frac{1}{2} *$ $\begin{array}{ccc} + & - & + \end{array}$ $\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ -1 & -2 & 0 \end{bmatrix}$ $= \frac{1}{2} * |(0-2)\hat{i} - (0-1)\hat{j} + (-2-1)\hat{k}|$

let $\vec{b} = \vec{RP} = \langle -1, -2, 0 \rangle$

↳ $\frac{1}{2}|-2\hat{i} + \hat{j} - 3\hat{k}| \rightarrow \frac{1}{2}\sqrt{4+1+9} = \frac{\sqrt{14}}{2}$

Q6 Determine any point of Intersection of $\quad L_1: x = 3+4t \quad L_2: x = -8+s$

$x=x, \; y=y, \; z=z$ $\quad y = 3-t \quad y = -4+3s$

$z = 3-t \quad z = 2+s$

$y=y \rightarrow 3-t = 3s-4 \quad \therefore 2+\cancel{s} = 3s-\cancel{4} \rightarrow 2s = 6 \rightarrow \boxed{s=3}$
$\quad {}_{+4\;-\cancel{s}} \quad {}_{-s\;\cancel{+4}}$

$z=z \rightarrow 3-t = 2+s$ $\quad$ ↳ $3-t = 2+s \rightarrow \cancel{3}-t = 5 \rightarrow -t = 2 \rightarrow \boxed{t=-2}$
$\quad {}_{-3} \quad {}_{-3}$

↳ In L_1 ($t=2$) $\quad x = 3+-8 = -5$

$y = 3--2 = 5 \quad (-5, 5, 5)$

$z = 3--2 = 5$

In L_2 ($s=3$) $\quad x = -8+3 = -5$

$y = -4+9 = 5 \quad (-5, 5, 5)$

$z = 2+3 = 5$

∴ The point of intersection is $(-5, 5, 5)$

Q7 Determine wether the line and plane intersect and find the point of Intersection.

$L: x = 1+t \quad y = -1+3t \quad z = 2+4t \qquad \pi: x - y + 4z = 7$

A.) Are they parallel? L vector: $\langle 1,3,4 \rangle$ $\quad \vec{n}$ of the plane: $\langle 1,-1,4 \rangle$ $\quad \vec{L}\cdot\vec{n} = 1-3+16 \neq 0$ $\quad \therefore$ Not parallel

B) Intersection: Substitude x,y,z from L_1 into the plane equation $\rightarrow \pi: 1+t+1-3t+8+16t=7 \rightarrow 14t=-3 \rightarrow \boxed{t=\frac{-3}{14}}$

$\therefore$ In L_1: $x = 1-\frac{3}{14} = \frac{11}{14}$

$y = \frac{-9}{14} - 1 = \frac{-23}{14} \longrightarrow \therefore$ The point of Intersection is $\left(\frac{11}{14}, \frac{-23}{14}, \frac{8}{7}\right)$

$z = 2 - \frac{12}{14} = \frac{16}{14} = \frac{8}{7}$

Q8 Find the distance between:

I. $p(1,-2,3)$ and $\pi: 2x-2y+z=4$

* The formula of the Distance between a point and a plane is $\boxed{D = \frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}}$ $\quad \therefore D = \frac{|2+4+3-4|}{\sqrt{4+4+1}} = \frac{5}{3}$

II. $\pi_1: -2x+y+z=0$ and $\pi_2: 6x-3y-3z-5=0$

* check if they are parallel $\left(\frac{n_1x}{n_2x} = \frac{n_1y}{n_2y} = \frac{n_1y}{n_2y}\right)$ $\rightarrow \frac{-2}{6} = \frac{1}{-3} = \frac{1}{-3}$ ✓ $\quad \#$ parallel

* take a point. let $y=z=0$ in π_1, $x=0$ $\therefore (0,0,0)$

Distance p_0 and plane π_2

$D = \frac{|0+0+0-5|}{\sqrt{36+9+9}} = \frac{5}{\sqrt{54}} = \frac{5}{3\sqrt{6}}$

Q9 show that $L_1: x=-1+t$ and $\pi: 2x-2y-2z+3=0$ are parallel and find the Distance between them.

$\vec{d}\langle 1,2,-1 \rangle$ $\quad y = 3+2t$ $\qquad \vec{n}\langle 2,-2,-2 \rangle$

$z = -t$

* check if parallel. $\vec{d}\cdot\vec{n} = 0 \rightarrow 2-4+2=0$ $\quad \therefore$ parallel

* Distance. lets assume $t=0$ $\quad L_1(-1,3,0)$ $\quad D = \frac{|-2-6-0+3|}{\sqrt{4+4+4}} = \frac{5}{\sqrt{12}} = \frac{5}{2\sqrt{3}}$

Distance between p_0 and plane

Q10 L_1: $x = -1+5t$, $y = 2+2t$, $z = 1-3t$

L_2: $x = 1-4s$, $y = 1+2s$, $z = 2-2s$

I. Show that they're Intersected and find the point of Intersection.

$x = x \rightarrow \boxed{5t-1 = 1-4s}$ ①

$y = y \rightarrow 2+2t = 1+2s \xrightarrow{\times 2} \boxed{4+4t = 2+4s}$ ②

① ② $\rightarrow 4t+3 = 3 \rightarrow \boxed{t=0}$

In ② $\therefore 4 = 2+4s \rightarrow \boxed{s = \frac{1}{2}}$

$\rightarrow L_1 \overset{t=0}{:} (-1,2,1)$ $\quad L_2 \overset{s=\frac{1}{2}}{:} (-1,2,1) \rightarrow \therefore$ Intersected at $(-1,2,1)$

II. Find the plane containing the intersecting lines.

* The plane must have $\vec{n}$ perpendicular to the direction of L_1, L_2 * $\vec{n} = \vec{d_1} \times \vec{d_2}$

$\vec{d_1} = \langle 5,2,-3 \rangle$ $\quad \vec{d_2} = \langle -4,2,-2 \rangle$

$$\rightarrow \vec{n} = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 2 & -3 \\ -4 & 2 & -2 \end{bmatrix} \end{array} = (-4+6)\hat{i} - (-10-12)\hat{j} + (10+18)\hat{k}$$

$\rightarrow n = \langle 2, -22, 18 \rangle$

* to find plane equation we're gonna use the intersection point and the $\vec{n}$

$2(x+1) - 22(y-2) + 18(z-1) = 0 \rightarrow 2x+2-22y+44+18z-18 \rightarrow 2x-22y+18z+28=0$

The End.